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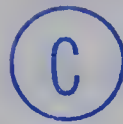
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THE UNIVERSITY OF ALBERTA

VIBRATION OF TAPERED CANTILEVER BEAMS

by



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A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES
IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE
OF MASTER OF SCIENCE

DEPARTMENT OF MECHANICAL ENGINEERING

EDMONTON, ALBERTA

JUNE, 1967



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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "Vibration of Tapered Cantilever Beams" submitted by JOHN G. RAE in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

Galerkin's method was used to calculate the natural frequencies of vibration of tapered cantilever beams. It was found that the values obtained from Galerkin's method converged very quickly, and for small tapers, reliable values for the fundamental and second natural frequencies were obtained when two and three terms were used in the trial solution.

The trial solution which was used in this thesis was found to be unsuitable for calculating natural frequencies of modes higher than the third mode, and for calculating the fundamental frequency of vibration when the effects of shearing force and rotatory inertia were considered.

Tests were conducted to determine the natural frequencies of vibration, and it was found that the experimental frequencies were always lower than the theoretical frequencies. The largest differences were found for the shorter beams, and the beams with the largest thickness taper.

From the theoretical values which were obtained for the second natural frequency, it was noted that for small tapers, the frequency of vibration decreased almost linearly, as the thickness taper was increased. This fact was used to propose a method for finding the second natural frequency of vibration of a beam, by using a linear extrapolation.

ACKNOWLEDGEMENTS

The author wishes to extend his appreciation to each of the following who contributed to the thesis:

- Dr. D.G. Bellow for supervising the thesis, and for his helpful suggestions regarding the experimental procedure and writing of the thesis,
- The staff members of the Mechanical Engineering Department who were responsible for the experimental apparatus,
- Miss L. Fiveland for her excellent typing of the thesis.

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NOTATION

y	beam deflection
x	distance along the beam from the fixed end
l	length of the beam
$X = \frac{x}{l}$	
α, β	width and thickness parameters
E	Young's modulus
G	shear modulus
ρ	mass density
I	moment of inertia of the cross section
A	area of the cross section
k	numerical factor which depends on the shape of the cross section
ω	frequency of vibration, cps
p	fundamental frequency of vibration of a uniform beam, cps
y_i	trial solution which contains i terms
ϕ_i	i^{th} shape function

Subscript o denotes values at the clamped end of the beam.

CHAPTER I

INTRODUCTION

1.1 Aim of Thesis

This thesis is concerned with determining the natural frequencies of vibration of straight, tapered cantilever beams of rectangular cross section. Galerkin's method [9,10]* will be used to determine the first three natural frequencies of vibration for linearly tapered cantilever beams when the effects of shearing force, rotatory inertia and damping are neglected. Galerkin's method will also be used to find the fundamental frequency of vibration of a beam of constant width and linearly varying thickness when the effects of shearing force and rotatory inertia are considered. Finally this method will be applied to the problem of finding the fundamental frequency of vibration of a cantilever beam with nonlinear tapers, when the effects of shearing force, rotatory inertia and damping are neglected.

For the special cases of a uniform beam and linearly tapered beams which are tapered to zero in one or both directions about the axis of symmetry, closed form solutions have been found [1,2,3]. The frequencies obtained from

* Numbers in brackets denote references contained in the Bibliography.

Galerkin's method will be compared to the frequencies obtained from the closed form solutions and from Martin's method [8] for these special cases.

Experiments will be conducted with several tapered cantilever beams to study their behavior when they are subjected to a cyclic disturbance. The data obtained from these experiments will be used to plot frequency response curves from which experimental values for the natural frequencies of vibration will be obtained.

1.2 Historical Introduction

The equation which governs the motion of a vibrating beam when the effects of shearing force, rotatory inertia and damping are neglected is

$$\frac{\partial^2}{\partial x^2} \left[EI \frac{\partial^2 y}{\partial x^2} \right] = -\rho A \frac{\partial^2 y}{\partial t^2} . \quad (1.2.1)$$

When considering a prismatic beam, this equation simplifies to

$$EI \frac{\partial^4 y}{\partial x^4} = -\rho A \frac{\partial^2 y}{\partial t^2} ,$$

which is easily solved for the natural frequencies of vibration when the boundary conditions are applied [1]. For the case

of a cantilever beam the boundary conditions are:

$$\begin{aligned} y(0,t) &= 0 \\ y'(0,t) &= 0 \\ y''(l,t) &= 0 \\ y'''(l,t) &= 0. \end{aligned}$$

In 1877, Rayleigh considered the effect of rotatory inertia and added a term to the right hand side of equation (1.2.1) to account for the added inertia of the beam, and in 1921 Timoshenko considered the effect of a shearing force on the motion of a vibrating beam. When these two effects are considered, the equation governing the motion of a prismatic beam becomes

$$EI \frac{\partial^4 y}{\partial x^4} + \rho A \frac{\partial^2 y}{\partial t^2} - \left(\rho I + \frac{\rho EI}{k' G} \right) \frac{\partial^4 y}{\partial x^2 \partial t^2} + \frac{\rho^2 I}{k' G} \frac{\partial^4 y}{\partial t^2} = 0.$$

One of the earliest methods used to determine the natural frequency of a tapered cantilever beam was applied by Kirchoff in 1879, when the problem of a vibrating cone with a clamped base and free tip was considered. Kirchoff found an infinite series solution for the resulting differential equation, which gave the fundamental frequency of vibration of the cone.

In 1913 Ward [2] extended Kirchoff's method to include bars of rectangular cross section. Ward considered the case of a beam with an area of zero at the free end, and used

the free end of the beam as the origin of the co-ordinates.
(Figure 1-1)

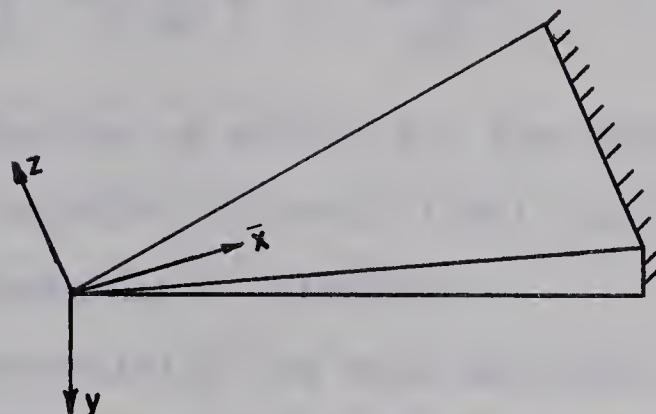


FIG. 1-1 WARD'S BEAM

Ward let the width vary as $\bar{x}^n$ and the thickness vary as $\bar{x}^m$, so that the area and moment of inertia of the cross section could be expressed as

$$A = A_0 \bar{x}^{n+m}$$

$$I = I_0 \bar{x}^{n+3m},$$

where A_0 and I_0 are the area and moment of inertia of the cross section at $\bar{x} = 1$. By separating the variables in the differential equation

$$\frac{\partial^2}{\partial X^2} \left[EI \frac{\partial^2 y}{\partial X^2} \right] = -\rho A \frac{\partial^2 y}{\partial t^2},$$

and using the substitutions (for the case of $m = 1$)

$$\eta^2 = 4\bar{x} \sqrt{\frac{\rho \omega^2 A_0}{EI_0}}$$

and

$$\tau = y \eta^{n+1},$$

Ward found

$$\frac{d^2\tau}{d\eta^2} + \frac{1}{\eta} \frac{d\tau}{d\eta} + \tau - \frac{(n+1)^2}{\eta^2} \tau = 0, \quad (1.2.2)$$

to be the equation of motion for the beam. Equation (1.2.2) is a Bessel equation of order $(n+1)$, and its roots give the natural frequencies of vibration.

Using essentially the same approach, Wrinch [3] in 1922, arrived at the same results as Ward and Kirchoff.

In 1962, Housner and Knightley [4] studied the vibrations of a linearly tapered cantilever beam by using a stepped beam. They considered the beam to be made up of a finite number of prismatic sections, and wrote the following equations for the shear, bending moment, slope, and deflection for each section.

$$V_n = V_0 + \sum_{j=1}^{n-1} m_j y_j \omega^2,$$

$$M_n = M_0 + \sum_{j=1}^n V_j \Delta x_j,$$

$$\theta_n = \theta_0 + \sum_{j=1}^n \frac{M_j + M_{j-1} \Delta x_j}{2EI_j}$$

$$Y_n = Y_{n-1} + \theta_{n-1} \Delta x_n + \frac{M_{n-1} (\Delta x_n)^2}{3EI_n} + \frac{M_n (\Delta x_n)^2}{6EI_n}$$

To determine the natural frequency from these equations, Housner and Knightley started with an approximate mode shape

and used successive iterations to improve the mode shape until all of the boundary conditions were satisfied. When the boundary conditions were satisfied, the corresponding value of ω was calculated.

In 1963, Leckie and Lindberg [5] and Lindberg [6] modified the method of using lumped parameters to determine the frequencies of vibration of linearly tapered cantilever beams. By assuming the deflected shape of each element to be a polynomial of the form

$$\bar{y} = a + bx + cx^2 + dx^3,$$

Leckie and Lindberg were able to write a dynamic stiffness matrix which related the shear and bending moment to the slopes and deflections of the ends of the elements. When the boundary conditions were satisfied, the parameters a , b , c , and d could be determined and the natural frequencies calculated.

An attempt to apply finite difference equations to the differential equations that govern the motion of a linearly tapered cantilever beam when the effects of shearing force and rotatory inertia are considered was made by Rissoné and Williams [7] in 1965. By replacing the derivatives in the equations of motion

$$\frac{\partial}{\partial x} \left[EI \frac{\partial \psi}{\partial x} \right] + k' \left[\frac{\partial y}{\partial x} - \psi \right] AG = + \rho I \frac{\partial^2 \psi}{\partial t^2},$$

$$\frac{\partial}{\partial x} \left[k' AG \left(\frac{\partial y}{\partial x} - \psi \right) \right] = \rho A \frac{\partial^2 y}{\partial t^2},$$

by the corresponding finite difference expressions and adding the equations together, Rissoné and Williams were able to write one equation in the two variables Y and ψ for each element of the beam. For a beam made up of n elements this procedure gave n simultaneous equations which were solved for the natural frequencies of vibration.

Another approach to the problem of determining the natural frequencies of a tapered beam is to treat the beam as a single unit. In 1956, Martin [8] considered a linearly tapered cantilever beam with the origin of the co-ordinate system at the clamped end, and expressed the area and moment of inertia of the cross section as

$$A = A_0 f(x, \alpha, \beta),$$

$$I = I_0 g(x, \alpha, \beta),$$

where α and β are the width and thickness parameters, and A_0 and I_0 are the area and moment of inertia at the clamped end. The equation of motion then becomes

$$\frac{d^2}{dx^2} \left[g(x, \alpha, \beta) \frac{d^2 y}{dx^2} \right] = \lambda f(x, \alpha, \beta) y,$$

where

$$\lambda = \frac{A_0 \rho \omega^2}{EI_0}.$$

Martin assumed that

$$\lambda = \lambda_0 + \lambda_1 \alpha + \lambda_2 \beta + \lambda_3 \alpha \beta + \lambda_4 \alpha^2 + \lambda_5 \beta^2 + \dots,$$

and

$$y = y_0 + y_1 \alpha + y_2 \beta + y_3 \alpha \beta + y_4 \alpha^2 + y_5 \beta^2 \dots,$$

where λ_i 's are parameters to be determined and y_i 's are functions of x . By considering beams with small tapers, ($\alpha \leq 0.5$, $\beta \leq 0.5$) terms of greater than second order in α and β become small and were neglected. Substituting the functions for λ and y into the differential equation for motion, and comparing like powers of α and β a series of differential equations was formed, which could be solved for the values of λ_i . Using these values of λ_i the natural frequencies of vibration were calculated. This method gave a lower bound for the natural frequencies of vibration.

In 1965, Rao [9] found that by applying Galerkin's method to the equation of motion for a vibrating beam,

$$\frac{\partial^2}{\partial x^2} \left[EI \frac{\partial^2 y}{\partial x^2} \right] = -\rho A \frac{\partial^2 y}{\partial t^2} ,$$

acceptable values could be obtained for the fundamental frequency of linearly tapered beams. Rao's values were an upper bound to the actual frequencies, and were obtained by solving only one equation. The details of this method are given in Chapter II.

1.3 Summary

The problem of finding the natural frequencies of vibration of non-uniform cantilever beams is the same problem

as determining the natural frequencies of vibration of turbine blades. This problem has attracted the attention of many investigators, and for the special cases of a uniform beam and linearly tapered beams which have an area of zero at the free end, closed form solutions are available. However, in most applications the beams are not uniform, nor do they have an area of zero at the free end. When this is the case, approximate methods have to be used to determine the natural frequencies of vibration.

When the problem is analyzed using a finite difference method, of the type used by Rissoné and Williams, or a lumped parameter method, of the type used by Housner and Knightley, and Leckie and Lindberg, each element of the beam must be considered separately, and the number of equations which are formed is equal to the number of elements considered. To solve these equations, it is usual to use a digital computer, and the accuracy of the results depends on the number of elements used. For tapered beams the results converge slowly, and as a result a large number of elements have to be considered. The results obtained using these methods are always higher than the experimental values, and the values calculated using the closed form solutions.

When the problem is considered by treating the beam as a continuous unit, as done by Martin and Rao, it was found that the number of equations which have to be solved in order

to determine the frequencies is much less than the number of equations which have to be solved for the finite difference and lumped parameter methods. As a result, a desk calculator can be used instead of a digital computer. In view of this fact, it appears that these methods of solution are more practical than the finite difference and lumped parameter methods. However, on surveying the literature available, it was found that Galerkin's method has only been used to find the fundamental frequency of vibration of linearly tapered cantilever beams, and before any judgement can be made on the actual merit of Galerkin's method as a means of determining the natural frequencies of vibration, it must be applied to a wider range of problems.

In Chapter II, Galerkin's method is used to determine the first three natural frequencies of vibration of linearly tapered cantilever beams. It is also applied to the problem of determining the fundamental frequency of a beam which has a constant width and a linearly varying thickness, when the effects of shearing force and rotatory inertia are considered. Finally Galerkin's method is used to determine the fundamental frequency of vibration of a cantilever beam with nonlinear tapers. The convergence of the results is discussed and the results are compared to the results which were obtained from the closed form solution and Martin's solution.

CHAPTER II

THEORETICAL ANALYSIS AND RESULTS

2.1 Plan of Chapter

In Section 2.2, the equations of motion for vibrating cantilever beams are formulated when the effects of taper, shearing force and rotatory inertia are considered. The equations for a uniform beam with and without the effects of shearing force and rotatory inertia are presented with their closed form solutions in Subsection 2.2-1. In Subsection 2.2-2 the equations of motion which were used to determine the natural frequencies of vibration of tapered beams are presented.

In Section 2.3 Galerkin's method is used to calculate the natural frequencies of vibration of tapered cantilever beams. Subsection 2.3-1 contains a brief discussion of the requirements of Galerkin's method, and in Subsection 2.3-2 the one, two, and three term trial solutions which were used in the application of Galerkin's method are derived. The three trial solutions are applied to the equation of motion of a linearly tapered beam, and the resulting equations which must be solved to determine the natural frequencies of vibration are given in Subsection 2.3-3. The one term trial solution is applied to the equation for a beam of constant

width and linearly varying thickness when the effects of shearing force and rotatory inertia are considered, and to the equation for a nonlinearly tapered beam in Subsections 2.3-4 and 2.3-5 respectively.

The results obtained for the natural frequencies are presented in Section 2.4. In Subsection 2.4-1 the convergence of Galerkin's is illustrated by comparing the frequencies obtained when the one, two, and three term trial solutions were applied to the equation of motion for a linearly tapered cantilever beam to the frequencies obtained from the closed form solutions. In Subsection 2.4-2 the frequencies obtained from Galerkin's method for the first three modes of linearly tapered cantilever beams are presented and discussed. The frequencies obtained when shearing force and rotatory inertia are considered are presented in Subsection 2.4-3, and the fundamental frequencies of beams with nonlinear tapers are given in Subsection 2.4-3.

In Section 2.5 the theoretical results and the problems encountered in calculating them are discussed.

2.2 The Equations of Motion

In general, it is assumed that the beam material is homogeneous and isotropic, therefore the mass density ρ , Young's modulus E , and the shear modulus G are constant.

The clamped end of the beam is the origin of the co-ordinate system and the positive direction is taken downward (See Figure 2-1). At the origin, the area and moment of inertia of the cross section are given by A_0 and I_0 respectively. The effect due to damping is neglected.

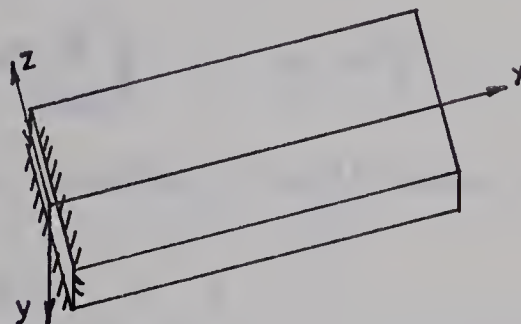


FIG. 2-1 CO-ORDINATE SYSTEM

2.2-1 The Uniform Beam

The equation of motion for a vibrating uniform beam can be shown to be

$$EI \frac{\partial^4 Y}{\partial x^4} = -\rho A \frac{\partial^2 Y}{\partial t^2}, \quad (2.2.1)$$

when the effects of shearing force and rotatory inertia are neglected. When the clamped end of the beam is taken at the origin of the co-ordinate system, the boundary conditions for a cantilever beam are

$$\begin{aligned} Y(0,t) &= 0 \\ Y'(0,t) &= 0 \\ Y''(\ell,t) &= 0 \\ Y'''(\ell,t) &= 0, \end{aligned}$$

where primes denote differentiation with respect to x .

By assuming Y to be of the form

$$Y(x,t) = y(x) \sin \omega t ,$$

Equation (2.2.1) reduces to

$$EI \frac{d^4 y}{dx^4} = \rho A \omega^2 y , \quad (2.2.2)$$

and the corresponding boundary conditions are

$$\begin{aligned} y(0) &= 0 \\ y'(0) &= 0 \\ y''(\ell) &= 0 \\ y'''(\ell) &= 0 . \end{aligned} \quad (2.2.3)$$

The general solution of Equation (2.2.2) is

$$\begin{aligned} y(x) &= C_1 \cosh(kx) + C_2 \sinh(kx) + C_3 \cos(kx) \\ &\quad + C_4 \sin(kx) , \end{aligned}$$

$$\text{where } k^4 = \frac{\omega^2 \rho A}{EI} .$$

When the boundary conditions (2.2.3) are satisfied the characteristic frequency equation becomes

$$\cos(k\ell) \cosh(k\ell) = -1 , \quad (2.2.4)$$

and the solution of the equation of motion becomes

$$\begin{aligned} Y(x,t) &= C_0 \left[(\cosh(kx) - \cos(kx)) - \frac{\cosh(k\ell) + \cos(k\ell)}{\sinh(k\ell) + \sin(k\ell)} \right. \\ &\quad \left. (\sinh(kx) - \sin(kx)) \right] \sin(\omega t) . \end{aligned}$$

The first three roots of Equation (2.2.4) are $\left[1 \right]$:

$$k_1 \ell = 1.875 = n_1$$

$$k_2 \ell = 4.694 = n_2$$

$$k_3 \ell = 7.855 = n_3 ,$$

and the corresponding natural frequencies of vibration are:

$$p_i = \left(\frac{n_i}{\ell} \right)^2 \sqrt{\frac{EI}{\rho A}} . \quad (2.2.5)$$

Shearing force and rotatory inertia are two effects which reduce the natural frequency of vibration. When shearing force is considered in the calculation of beam deflections, the deflections increase, which means that the stiffness of the beam is reduced and there is a corresponding reduction in the natural frequency. Rotatory inertia is present because, as the elements of the beam translate they also rotate, and in the case of a vibrating system the rotation of the beam adds inertia to the system. This additional inertia has the effect of increasing the mass of the system and results in a further reduction of the natural frequency of vibration.

When considering the combined effects of shearing force and rotatory inertia, the slope of the beam is assumed to consist of two components $\left[1 \right]$: ψ , which is the slope when shearing force is neglected, and ξ which is the angle of shear.

$$\frac{dy}{dx} = \psi + \xi .$$

The inertial moment caused by the angular acceleration of the beam is

$$dM_i = \rho I \frac{\partial^2 \psi}{\partial t^2} dx .$$

When shearing force is considered, the shear on the face of any element becomes

$$V = k' \xi AG$$

where k' is a numerical factor which depends on the shape of the cross section.

Considering these new terms, the equation for the rotation of the beam becomes

$$\frac{\partial}{\partial x} \left[EI \frac{\partial \psi}{\partial x} \right] + k' \left[\frac{\partial y}{\partial x} - \psi \right] AG = \rho I \frac{\partial^2 \psi}{\partial t^2} , \quad (2.2.6)$$

and the equation for the translation of the beam becomes

$$\frac{\partial}{\partial x} \left[k' AG \left[\frac{\partial y}{\partial x} - \psi \right] \right] = \rho A \frac{\partial^2 y}{\partial t^2} . \quad (2.2.7)$$

For the case of a uniform beam ψ can easily be eliminated from Equations (2.2.6) and (2.2.7), leaving the following equation in y .

$$EI \frac{\partial^4 y}{\partial x^4} + \rho A \frac{\partial^2 y}{\partial t^2} - \left(\rho I + \frac{\rho EI}{k' G} \right) \frac{\partial^4 y}{\partial x^2 \partial t^2} + \frac{\rho^2 I}{k' G} \frac{\partial^4 y}{\partial t^4} = 0 . \quad (2.2.8)$$

Solving Equation (2.2.8) and applying the boundary conditions for a cantilever beam gives the following frequency equation.

$$(A^4 + B^4) + 2A^2B^2 \cos(A\ell) \cosh(B\ell) + (A^3B - AB^3) \sin(A\ell) \sinh(B\ell) = 0 \quad (2.2.9)$$

where

$$A = \sqrt{\frac{a + \sqrt{a^2 - 4b}}{2}}, \quad B = \sqrt{\frac{-a + \sqrt{a^2 - 4b}}{2}},$$

$$a = \frac{\rho\omega^2}{E} + \frac{\rho\omega^2}{kG},$$

$$b = \frac{\rho^2\omega^4}{kEG} - \frac{\rho A\omega^2}{EI}.$$

Another case when ψ can be eliminated from Equations (2.2.6) and (2.2.7) is for a beam of constant width and linearly varying thickness. This case is discussed in Subsection (2.2-2).

2.2-2 The Tapered Beam

Consider the beam shown in Figure (2-2), which has a rectangular cross section and length ℓ . Let it be clamped at the large end and free at the small end.

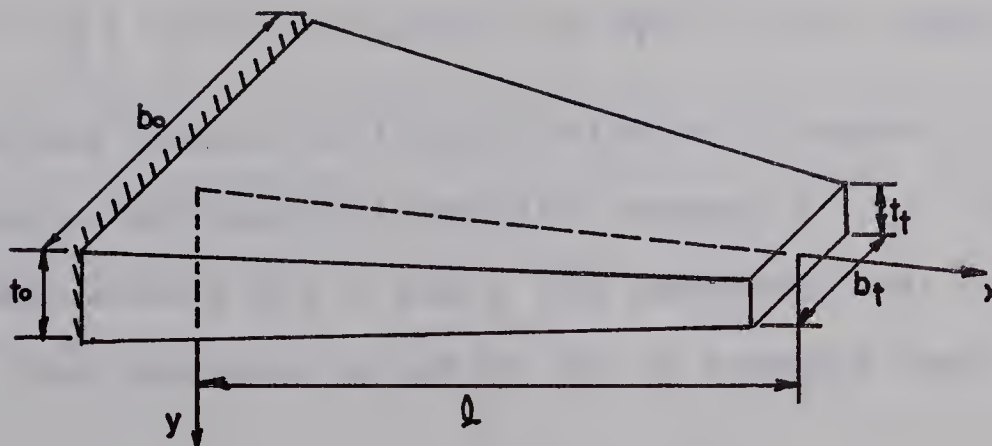


FIG. 2-2 TAPERED BEAM

Defining the taper parameters as

$$\alpha = \frac{b_0 - b_t}{b_0}$$

and
$$\beta = \frac{t_0 - t_t}{t_0},$$

the width and thickness of the beam at any point a distance x from the origin is

$$b = b_0 \left(1 - \alpha \left(\frac{x}{\ell} \right)^n \right),$$

$$t = t_0 \left(1 - \beta \left(\frac{x}{\ell} \right)^m \right).$$

Defining I_0 and A_0 as the moment of inertia and the area of the cross section at the clamped end, the moment of inertia and area at any point in the beam may be expressed as

$$I = I_0 (1 - \alpha X^n) (1 - \beta X^m)^3, \quad (2.2.10)$$

$$A = A_0 (1 - \alpha X^n) (1 - \beta X^m), \quad (2.2.11)$$

where $X = \frac{x}{\ell}$.

For a non uniform beam, the equation of motion is

$$E \left[I'' Y'' + 2I' Y''' + IY^{IV} \right] = - \rho A \ddot{Y},$$

where primes denote differentiation with respect to x and dots denote differentiation with respect to t . Using the above expressions for I and A and assuming that $Y(X,t) = y(x) \sin \omega t$, the equation of motion for a tapered beam becomes

$$\begin{aligned}
\frac{1}{\ell^2} \left[-n(n-1)\alpha X^{n-2}(1-\beta X^m)^3 + 6mn\alpha\beta X^{m+n-2}(1-\beta X^m)^2 \right. \\
\left. - 3m(m-1)\beta X^{m-2}(1-\alpha X^n)(1-\beta X^m)^2 + 6m^2\beta^2 X^{2m-2} \right. \\
\left. (1-\alpha X^n)(1-\beta X^m) \right] Y'' - \frac{2}{\ell} \left[n\alpha X^{n-1}(1-\beta X^m)^3 \right. \\
\left. + 3m\beta X^{m-1}(1-\alpha X^n)(1-\beta X^m)^2 \right] Y''' + (1-\alpha X^n) \\
(1-\beta X^m)^3 Y^{IV} = \frac{\omega^2 \rho A_0}{EI_0} (1-\alpha X^n)(1-\beta X^m) Y. \quad (2.2.12)
\end{aligned}$$

For a linear taper, n and m are both equal to one, and Equation (2.2.12) simplifies to

$$\begin{aligned}
(1-\alpha X)(1-\beta X)^3 Y^{IV} - \frac{2}{\ell} \left[\alpha(1-\beta X)^3 + 3\beta(1-\alpha X)(1-\beta X)^2 \right] Y''' \\
+ \frac{1}{\ell^2} \left[6\beta^2(1-\alpha X)(1-\beta X) + 6\alpha\beta(1-\beta X)^2 \right] Y'' = \frac{\omega^2 \rho A_0}{EI_0} \\
(1-\alpha X)(1-\beta X) Y. \quad (2.2.13)
\end{aligned}$$

For the limiting cases of a wedge ($\alpha = 0, \beta = 1$) [2,3] and a cone ($\alpha = \beta = 1$) [3], Equation (2.2.13) can be reduced to a Bessel equation, which can be solved for the natural frequencies of vibration. However, there is no general analytical solution to Equation (2.2.13).

When the effects of shearing force and rotatory inertia are considered in the problem of determining the natural frequencies of vibration of tapered cantilever beams, it becomes very difficult to eliminate either Y or ψ from the equations

$$\frac{\partial}{\partial x} \left[EI \frac{\partial \psi}{\partial x} \right] + k' \left[\frac{\partial Y}{\partial x} - \psi \right] AG - \rho I \frac{\partial^2 \psi}{\partial t^2} = 0 \quad (2.2.6)$$

$$\rho A \frac{\partial^2 Y}{\partial t^2} - k' \frac{\partial}{\partial x} \left[\left(\frac{\partial Y}{\partial x} - \psi \right) AG \right] = 0. \quad (2.2.7)$$

One case for which ψ has been eliminated is for a wedge*.

The procedure used to eliminate ψ is outlined in the following discussion.

Since

$$\frac{dy}{dx} = \psi + \xi$$

and $Y(x,t) = y(x) \sin(\omega t)$, $\psi(x,t)$ can be assumed to be

$$\psi(x,t) = \psi(x) \sin(\omega t).$$

Substituting these values for $Y(x,t)$ and $\psi(x,t)$ into Equations 2.2.6 and 2.2.7 produces the two coupled ordinary differential equations

$$\frac{d}{dx} \left[EI \frac{d\psi}{dx} \right] + k' \left[\frac{dy}{dx} - \psi \right] AG = -\rho \omega^2 I \psi \quad (2.2.14)$$

$$\frac{d}{dx} \left[k' AG \left(\frac{dy}{dx} - \psi \right) \right] = -\rho \omega^2 A y \quad (2.2.15)$$

For a wedge, ($\alpha = 0$) the area and moment of inertia of the cross section is given by

$$A = A_0 (1 - \beta X),$$

$$I = I_0 (1 - \beta X)^3,$$

where $X = \frac{x}{l}$.

* The term "wedge" is used here to describe a beam of constant width ($\alpha = 0$) and linearly varying thickness ($\beta = \text{constant}$).

Substituting these values for A and I into Equations (2.2.14) and (2.2.15) give the following equations for the rotation and translation of the wedge

$$\frac{d}{dx} \left[E(1-\beta X)^3 \frac{d\psi}{dx} \right] + D(1-\beta X) \left(\frac{dy}{dx} - \psi \right) = -\rho\omega^2(1-\beta X)^3\psi, \quad (2.2.16)$$

$$\frac{d}{dx} \left[(1-\beta X) \left(\frac{dy}{dx} - \psi \right) \right] = -\frac{\rho\omega^2}{k' G} (1-\beta X)y, \quad (2.2.17)$$

where $D = \frac{k' GA_0}{I_0}$.

To simplify these equations, let

$$(1 - \beta X) = u,$$

then $d/dx = -(\beta/\ell)(d/du)$.

Substitute u into Equations (2.2.16) and (2.2.17) and differentiate (2.2.16) with respect to u to get the following

$$\begin{aligned} c^2 E \left[6u \psi' + 6u^2 \psi'' + u^3 \psi''' \right] + \frac{D\rho\omega^2}{ck' G} uy &= -\rho\omega^2 \\ &\left[3u^2 \psi + u^3 \psi' \right], \\ (cy' + \psi) + u(cy'' + \psi') &= -\frac{\rho\omega^2}{ck' G} uy, \end{aligned}$$

where $c = (\beta/\ell)$, and the primes represent differentiation with respect to u. By employing a series of differentiations and substitutions, ψ can be eliminated and the resulting equation of motion becomes

$$\frac{1}{c\ell^4} \frac{d^4 y}{dx^4} \left[E(1-\beta X)^2 (D - \rho\omega^2(1-\beta X)^2 + Ec^2) \right]$$

$$\begin{aligned}
& - \frac{1}{\ell^3} \frac{d^3 y}{dX^3} \left[E(1-\beta X) (6D - 4\rho\omega^2(1-\beta X)^3 + 6Ec^2) \right] \\
& + \frac{1}{c\ell^2} \frac{d^2 y}{dX^2} \left[(D + Ec^2) (6Ec^2 + \rho\omega^2(1-\beta X)^2) - \rho^2\omega^4(1-\beta X)^4 \right. \\
& \left. + \frac{(D + Ec^2)}{k'G} E\rho\omega^2(1-\beta X)^2 - \frac{E\rho^2\omega^4}{k'G} (1-\beta X)^4 \right] \\
& - \frac{1}{\ell} \frac{dy}{dX} \left[(D + Ec^2) \left[3\rho\omega^2(1-\beta X) + \frac{5E\rho\omega^2}{k'G} (1-\beta X) \right] \right. \\
& \left. - \frac{3E\rho^2\omega^4}{k'G} (1-\beta X)^3 - \rho^2\omega^4 (1-\beta X)^3 \right] \\
& + y \left[\frac{(D + Ec^2)}{ck'G} \left[3Ec^2\rho\omega^2 - D\rho\omega^2 + 2\rho^2\omega^4 (1-\beta X)^2 \right] \right. \\
& \left. - \frac{\rho^3\omega^6}{ck'G} (1-\beta X)^4 \right] = 0 . \tag{2.2.18}
\end{aligned}$$

The details of the derivation of this equation are given in Appendix I-A.

Equation (2.2.18) is used in Subsection 2.3-4 to determine the effects of shearing force and rotatory inertia on fundamental frequency of a wedge.

2.3 Determination of the Natural Frequencies Using Galerkin's Method

2.3-1 Galerkin's Method

This method was proposed by Galerkin in 1915, as a means of finding an approximate solution to boundary value

problems [10].

The solution of the homogeneous differential equation

$$L(z) = a_0(x) \frac{d^n z}{dx^n} + a_1(x) \frac{d^{n-1} z}{dx^{n-1}} + \dots + a_{n-1}(x) \frac{dz}{dx} + a_n(x) z = 0 ,$$

can be approximated by the partial sum

$$z_n(x) = \sum_{i=1}^n c_i \phi_i(x) , \quad (2.3.1)$$

when the functions $\phi_i(x)$ are selected to satisfy the boundary conditions of the problem. The approximate solution $z_n(x)$, however does not exactly satisfy the differential equation $L(z) = 0$, and when $z_n(x)$ is used in the differential equation, an error is present. By selecting the constants c_i such that

$$\int_0^l L(z_n) \phi_i(x) dx = 0 \quad i = 1, 2, \dots, n$$

the error is minimized, and the expression (2.3.1) becomes an acceptable approximation to the solution.

2.3-2 Determination of the Trial Solutions

In order to apply Galerkin's method it becomes necessary to find a set of functions $\phi_i(x, t)$ such that the boundary conditions are satisfied. When the functions $\phi_i(x, t)$ are found, the motion of the beam can be approximated

by

$$Y_n(x,t) = \sum_{i=1}^n a_i \phi_i(x,t) .$$

Since $Y(x,t) = y(x) \sin(\omega t)$, $\phi_i(x,t)$ can be approximated by $\phi_i(x) \sin(\omega t)$. The boundary conditions which $\phi_i(x)$ must satisfy are

$$\phi_i(0) = 0$$

$$\phi_i'(0) = 0$$

$$\phi_i''(\ell) = 0$$

$$\phi_i'''(\ell) = 0 .$$

Let $y(x)$ be a polynomial of the form

$$y(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \dots + c_jx^j .$$

When the boundary conditions are satisfied with $j = 4$, $y_1(x)$ is found to be

$$y_1(x) = a_1(6x^2 - 4x^3 + x^4) ,$$

$$\text{and} \quad \phi_1(x) = (6x^2 - 4x^3 + x^4) , \quad (2.3.2)$$

for $j = 5$, $y_2(x)$ becomes

$$y_2(x) = a_1(6x^2 - 4x^3 + x^4) + a_2(20x^2 - 10x^3 + x^5) ,$$

$$\phi_1(x) = (6x^2 - 4x^3 + x^4) ,$$

$$\phi_2(x) = (20x^2 - 10x^3 + x^5) , \quad (2.3.3)$$

and for $j = 6$, $y_3(x)$ becomes

$$\begin{aligned} y_3(x) = & a_1(6x^2 - 4x^3 + x^4) + a_2(20x^2 - 10x^3 + x^5) \\ & + a_3(45x^2 - 20x^3 + x^6) , \end{aligned}$$

$$\begin{aligned}
\phi_1(X) &= (6X^2 - 4X^3 + X^4) , \\
\phi_2(X) &= (20X^2 - 10X^3 + X^5) , \\
\phi_3(X) &= (45X^2 - 20X^3 + X^6) ,
\end{aligned} \tag{2.3.4}$$

where $X = \frac{x}{\ell}$.

This process can be carried out indefinitely, and each new trial solution for the deflected form is slightly better than the previous approximation.

2.3-3 Natural Frequencies of Linearly Tapered Beams

The governing differential equation for a linearly tapered cantilever beam when the effects of shearing force and rotatory inertia are neglected is

$$\begin{aligned}
L[y] &= (1-\alpha X)(1-\beta X)^3 y^{IV} - \frac{2}{\ell} \left[\alpha(1-\beta X)^3 + 3\beta(1-\alpha X)(1-\beta X)^2 \right] \\
&\quad y''' + \frac{1}{\ell^2} \left[6\beta^2(1-\alpha X)(1-\beta X) + 6\alpha\beta(1-\beta X)^2 \right] y'' \\
&\quad - \frac{\omega^2 \rho A_0}{EI_0} (1-\alpha X)(1-\beta X) y = 0 ,
\end{aligned} \tag{2.3.5}$$

where primes denote differentiation with respect to x , and $X = \frac{x}{\ell}$.

Using the one term trial solution

$$y_1(X) = a_1(6X^2 - 4X^3 + X^4) , \tag{2.3.2}$$

in the application of Galerkin's method to Equation (2.3.5) gives the fundamental frequency of vibration when

$$\int_0^1 L[y_1] \phi_1 dx = 0.$$

After the substitutions are made for y_1 and the derivatives of y_1 this integral becomes

$$\begin{aligned} \int_0^1 & \left[\frac{12a_1}{l^4} \left[2(1-\alpha x)(1-\beta x)^3 + 4\alpha(1-\beta x)^3(1-x) \right. \right. \\ & + 12\beta(1-\alpha x)(1-\beta x)^2(1-x) + 6\beta^2(1-\alpha x)(1-\beta x)(1-2x+x^2) \\ & \left. \left. + 6\alpha\beta(1-\beta x)^2(1-2x+x^2) \right] - \frac{a_1 \omega^2 \rho A_0}{EI_0} (1-\alpha x)(1-\beta x)(6x^2-4x^3+x^4) \right] \\ & (6x^2-4x^3+x^4) dx = 0. \end{aligned}$$

When this integration is carried out, the fundamental frequency of vibration is found to be

$$\omega^2 = \frac{12EI_0}{\rho A_0 l^4} \left[\frac{\frac{12}{5} - (\alpha+3\beta)\frac{2}{5} + \frac{12}{35}(\alpha\beta+\beta^2) - \frac{3}{70}(3\alpha\beta^2+\beta^3) + \frac{2}{105}\alpha\beta^3}{\frac{104}{45} - \frac{584}{315}(\alpha+\beta) + \frac{5353}{3465}\alpha\beta} \right].$$

When the two term trial solution,

$$y_2(x) = a_1(6x^2-4x^3+x^4) + a_2(20x^2-10x^3+x^5), \quad (2.3.3)$$

is used in the application of Galerkin's method to Equation (2.3.5), two equations are formed.

$$\begin{aligned} & \int_0^1 L[y_2] \phi_1 dx \\ &= \int_0^1 \left[12a_4 \left[2(1-\alpha x)(1-\beta x)^3 + 12\beta(1-\alpha x)(1-\beta x)^2(1-x) \right. \right. \\ & \left. \left. + 4\alpha(1-\beta x)^3(1-x) + 6\beta^2(1-\alpha x)(1-\beta x)(1-2x+x^2) \right] \right. \end{aligned}$$

$$\begin{aligned}
& + 6\alpha\beta(1-\beta x)^2(1-2x+x^2) \Big] - \frac{\rho A_0 \omega^2 \ell^4 a_1}{EI_0} (1-\alpha x)(1-\beta x) \\
& (6x^2-4x^3+x^4) \Big] (6x^2-4x^3+x^4) dx \\
& + \int_0^1 \left[20a_2 \left[6(1-\alpha x)(1-\beta x)^3 x + 18\beta(1-\alpha x)(1-\beta x)^2(1-x^2) \right. \right. \\
& + 6\alpha(1-\beta x)^3(1-x^2) + 6\beta^2(1-\alpha x)(1-\beta x)(2-3x+x^3) \\
& + 6\alpha\beta(1-\beta x)^2(2-3x+x^3) \Big] - \frac{\rho A_0 \omega^2 \ell^4 a_2}{EI_0} (1-\alpha x)(1-\beta x) \\
& (20x^2-10x^3+x^5) \Big] (6x^2-4x^3+x^4) dx = 0
\end{aligned}$$

$$\begin{aligned}
& \int_0^1 L[y_2] \phi_2 dx \\
& = \int_0^1 \left[12a_1 \left[2(1-\alpha x)(1-\beta x)^3 + 12\beta(1-\alpha x)(1-\beta x)^2(1-x) \right. \right. \\
& + 4\alpha(1-\beta x)^3(1-x) + 6\beta^2(1-\alpha x)(1-\beta x)(1-2x+x^2) + 6\alpha\beta(1-\beta x)^2 \\
& (1-2x+x^2) \Big] - \frac{\rho A_0 \omega^2 \ell^4 a_1}{EI_0} (1-\alpha x)(1-\beta x)(6x^2-4x^3+x^4) \Big] \\
& (20x^2-10x^3+x^5) dx + \int_0^1 \left[20a_2 \left[6(1-\alpha x)(1-\beta x)^3 x \right. \right. \\
& + 18\beta(1-\alpha x)(1-\beta x)^2(1-x^2) + 6\alpha(1-\beta x)^3(1-x^2) + 6\beta^2 \\
& (1-\alpha x)(1-\beta x)(2-3x+x^3) + 6\alpha\beta(1-\beta x)^2(2-3x+x^3) \Big] \\
& - \frac{\rho A_0 \omega^2 \ell^4 a_2}{EI_0} (1-\alpha x)(1-\beta x)(20x^2-10x^3+x^5) \Big] (20x^2-10x^3+x^5) dx = 0 .
\end{aligned}$$

After the integrations are completed, the equations become

$$\begin{aligned}
 & a_1 \left[12 \left(\frac{12}{5} - \frac{2}{5}(\alpha+3\beta) + \frac{12}{35}(\alpha\beta+\beta^2) - \frac{3}{70}(3\alpha\beta^2+\beta^3) + \frac{2}{105} \alpha\beta^3 \right) \right. \\
 & \quad \left. - \frac{\omega^2 A_0 \rho \ell^4}{EI_0} \left(\frac{104}{45} - \frac{584}{315}(\alpha+\beta) + \frac{5353}{3465} \alpha\beta \right) \right] \\
 & + a_2 \left[20 \left(\frac{26}{5} - \frac{32}{35}(\alpha+3\beta) + \frac{57}{70}(\alpha\beta+\beta^2) - \frac{11}{105}(3\alpha\beta^2+\beta^3) + \frac{1}{21} \alpha\beta^3 \right) \right. \\
 & \quad \left. - \frac{\omega^2 A_0 \rho \ell^4}{EI_0} \left(\frac{2644}{315} - \frac{6679}{990}(\alpha+\beta) + \frac{8669}{1540} \alpha\beta \right) \right] = 0, \quad (2.3.6)
 \end{aligned}$$

$$\begin{aligned}
 & a_1 \left[12 \left(\frac{26}{3} - \frac{32}{21}(\alpha+3\beta) + \frac{19}{14}(\alpha\beta+\beta^3) - \frac{11}{63}(3\alpha\beta^2+\alpha^3) + \frac{5}{63} \alpha\beta^3 \right) \right. \\
 & \quad \left. - \frac{\omega^2 A_0 \rho \ell^4}{EI_0} \left(\frac{2644}{315} - \frac{6679}{990}(\alpha+\beta) + \frac{8669}{1540} \alpha\beta \right) \right] \\
 & + a_2 \left[20 \left(\frac{132}{7} - \frac{7}{2}(\alpha+3\beta) + \frac{68}{21}(\alpha\beta+\beta^2) - \frac{3}{7}(3\alpha\beta^2+\beta^3) + \frac{46}{231} \alpha\beta^3 \right) \right. \\
 & \quad \left. - \frac{\omega^2 A_0 \rho \ell^4}{EI_0} \left(\frac{21128}{693} - \frac{6187}{252}(\alpha+\beta) + \frac{184799}{9009} \alpha\beta \right) \right] = 0. \quad (2.3.7)
 \end{aligned}$$

In order to get a non-trivial solution to these two equations, the determinant of the coefficients of a_1 and a_2 must be set equal to zero. Writing Equations (2.3.6) and (2.3.7) as

$$\sum_{j=1}^2 (A_{ij} - k^2 B_{ij}) a_j = 0, \quad i = 1, 2,$$

where

$$A_{11} = 12 \left(\frac{12}{5} - \frac{2}{5}(\alpha+3\beta) + \frac{12}{35}(\alpha\beta+\beta^2) - \frac{3}{70}(3\alpha\beta^2+\beta^3) + \frac{2}{105} \alpha\beta^3 \right)$$

$$B_{11} = \left(\frac{104}{45} - \frac{584}{315}(\alpha+\beta) + \frac{5353}{3465} \alpha\beta \right)$$

$$A_{12} = 20 \left(\frac{26}{5} - \frac{32}{35}(\alpha+3\beta) + \frac{57}{70}(\alpha\beta+\beta^2) - \frac{11}{105}(3\alpha\beta^2+\beta^3) + \frac{1}{21} \alpha\beta^3 \right)$$

$$B_{12} = B_{21} = \frac{2644}{315} - \frac{6679}{990}(\alpha+\beta) + \frac{8669}{1540} \alpha\beta$$

$$A_{21} = 12 \left(\frac{26}{3} - \frac{32}{21}(\alpha+3\beta) + \frac{19}{14}(\alpha\beta+\beta^2) - \frac{11}{63}(3\alpha\beta^2+\beta^3) + \frac{5}{63} \alpha\beta^3 \right)$$

$$A_{22} = 20 \left(\frac{132}{7} - \frac{7}{2}(\alpha+3\beta) + \frac{68}{21}(\alpha\beta+\beta^2) - \frac{3}{7}(3\alpha\beta^2+\beta^3) + \frac{46}{231} \alpha\beta^3 \right)$$

$$B_{22} = \frac{21128}{693} - \frac{6187}{252}(\alpha+\beta) + \frac{184799}{9009} \alpha\beta$$

$$\kappa^2 = \frac{\rho A_0 \omega \ell^4}{EI_0},$$

it can be seen that $A_{21} = A_{12}$, and the determinant of the coefficients of a_1 and a_2 becomes

$$\begin{vmatrix} A_{11} - \kappa^2 B_{11} & A_{12} - \kappa^2 B_{12} \\ A_{12} - \kappa^2 B_{12} & A_{22} - \kappa^2 B_{22} \end{vmatrix}$$

When this determinant is expanded, and set equal to zero, the frequency equation becomes

$$\begin{aligned} (A_{11}A_{22} - A_{12}^2) - \kappa^2 (A_{11}B_{22} + A_{22}B_{11} - 2A_{12}B_{12}) + \kappa^4 (B_{11}B_{22} - B_{12}^2) \\ = 0. \end{aligned} \quad (2.3.8)$$

Equation (2.3.8) can readily be solved for two values of κ^2 from which the first two natural frequencies of vibration are determined.

When the three term trial solution,

$$y_3(x) = a_1(6x^2 - 4x^3 + x^4) + a_2(20x^2 - 10x^3 + x^5) + a_3(45x^2 - 20x^3 + x^6), \quad (2.3.4)$$

is used in the application of Galerkin's method to Equation (2.3.5), three equations are formed.

$$\begin{aligned} & \int_0^1 L[y_3] \phi_1 dx = \\ & \int_0^1 \left[12a_1 \left[2(1-\alpha x)(1-\beta x)^3 + 12\beta(1-\alpha x)(1-\beta x)^2(1-x) \right. \right. \\ & + 4\alpha(1-\beta x)^3(1-x) + 6\beta^2(1-\alpha x)(1-\beta x)(1-2x+x^2) + 6\alpha\beta \\ & \left. \left. (1-\beta x)^2(1-2x+x^2) \right] - \frac{\rho A_0 \omega^2 \ell^4 a_1}{EI_0} (1-\alpha x)(1-\beta x)(6x^2 - 4x^3 + x^4) \right] \\ & (6x^2 - 4x^3 + x^4) dx + \int_0^1 \left[20a_2 \left[6(1-\alpha x)(1-\beta x)^3 x + 18\beta(1-\alpha x) \right. \right. \\ & (1-\beta x)^2(1-x^2) + 6\alpha(1-\beta x)^3(1-x^2) + 6\beta^2(1-\alpha x)(1-\beta x) \\ & \left. \left. (2-3x+x^3) + 6\alpha\beta(1-\beta x)^2(2-3x+x^3) \right] - \frac{\rho A_0 \omega^2 \ell^4 a_2}{EI_0} \right. \\ & \left. (1-\alpha x)(1-\beta x)(20x^2 - 10x^3 + x^5) \right] (6x^2 - 4x^3 + x^4) dx + \int_0^1 \\ & \left[30a_3 \left[12(1-\alpha x)(1-\beta x)^3 x^2 + 24\beta(1-\alpha x)(1-\beta x)^2(1-x^3) \right. \right. \\ & + 8\alpha(1-\beta x)^3(1-x^3) + 6\beta^2(1-\alpha x)(1-\beta x)(3-4x+x^4) + 6\alpha\beta \end{aligned}$$

$$(1-\beta x)^2(3-4x+x^4) \Big] - \frac{\rho A_0 \omega^2 \ell^4 a_3}{EI_0} (1-\alpha x)(1-\beta x)(45x^2-20x^3+x^6) \Big]$$

$$(6x^2-4x^3+x^4) dx = 0.$$

$$\begin{aligned} & \int_0^1 L[y_3] \phi_2 dx \\ &= \int_0^1 \Big[12a_1 \Big[2(1-\alpha x)(1-\beta x)^3 + 12\beta(1-\alpha x)(1-\beta x)^2(1-x) \\ &+ 4\alpha(1-\beta x)^3(1-x) + 6\beta^2(1-\alpha x)(1-\beta x)(1-2x+x^2) + 6\alpha\beta \\ &(1-\beta x)^2(1-2x+x^2) \Big] - \frac{\rho A_0 \omega^2 \ell^4 a_1}{EI_0} (1-\alpha x)(1-\beta x)(6x^2-4x^3+x^4) \Big] \\ &(20x^2-10x^3+x^5) dx + \int_0^1 \Big[20a_2 \Big[6(1-\alpha x)(1-\beta x)^3x \\ &+ 18\beta(1-\alpha x)(1-\beta x)^2(1-x^2) + 6\alpha(1-\beta x)^3(1-x^2) + 6\beta^2(1-\alpha x) \\ &(1-\beta x)(2-3x+x^3) + 6\alpha\beta(1-\beta x)^2(2-3x+x^3) \Big] - \frac{\rho A_0 \omega^2 \ell^4 a_2}{EI_0} \\ &(1-\alpha x)(1-\beta x)(20x^2-10x^3+x^4) \Big] (20x^2-10x^3+x^4) dx \\ &+ \int_0^1 \Big[30a_3 \Big[12(1-\alpha x)(1-\beta x)^3x^2 + 24\beta(1-\alpha x)(1-\beta x)^2(1-x^3) \\ &+ 8\alpha(1-\beta x)^3(1-x^3) + 6\beta^2(1-\alpha x)(1-\beta x)(3-4x+x^4) + 6\alpha\beta \\ &(1-\beta x)^2(3-4x+x^4) \Big] - \frac{\rho A_0 \omega^2 \ell^4 a_3}{EI_0} (1-\alpha x)(1-\beta x)(45x^2-20x^3 \\ &+x^6) \Big] (20x^2-10x^3+x^5) dx = 0. \end{aligned}$$

$$\begin{aligned}
& \int_0^1 L[Y_3] \phi_3 dx \\
&= \int_0^1 \left[12a_1 \left[2(1-\alpha x)(1-\beta x)^3 + 12\beta(1-\alpha x)(1-\beta x)^2(1-x) \right. \right. \\
&+ 4\alpha(1-\beta x)^3(1-x) + 6\beta^2(1-\alpha x)(1-\beta x)(1-2x+x^2) + 6\alpha\beta \\
&\left. \left. (1-\beta x)^2(1-2x+x^2) \right] - \frac{\rho A_0 \omega^2 \ell^4 a_1}{EI_0} (1-\alpha x)(1-\beta x)(6x^2-4x^3+x^4) \right] \\
&(45x^2-20x^3+x^6) dx + \int_0^1 \left[20a_2 \left[6(1-\alpha x)(1-\beta x)^3 x \right. \right. \\
&+ 18\beta(1-\alpha x)(1-\beta x)^2(1-x^2) + 6\alpha(1-\beta x)^3(1-x^2) + 6\beta^2(1-\alpha x) \\
&\left. \left. (1-\beta x)(2-3x+x^3) + 6\alpha\beta(1-\beta x)^2(2-3x+x^3) \right] - \frac{\rho A_0 \omega^2 \ell^4 a_2}{EI_0} \right. \\
&\left. (1-\alpha x)(1-\beta x)(20x^2-10x^3+x^5) \right] (45x^2-20x^3+x^6) dx \\
&+ \int_0^1 \left[30a_3 \left[12(1-\alpha x)(1-\beta x)^3 x^2 + 24\beta(1-\alpha x)(1-\beta x)^2(1-x^3) \right. \right. \\
&+ 8\alpha(1-\beta x)^3(1-x^3) + 6\beta^2(1-\alpha x)(1-\beta x)(3-4x+x^4) + 6\alpha\beta \\
&\left. \left. (1-\beta x)^2(3-4x+x^4) \right] - \frac{\rho A_0 \omega^2 \ell^4 a_3}{EI_0} (1-\alpha x)(1-\beta x)(45x^2-20x^3+x^6) \right] \\
&(45x^2-20x^3+x^6) dx = 0.
\end{aligned}$$

After the integrations are completed, the equations become,

$$\begin{aligned}
& 12a_1 \left(\frac{12}{5} - \frac{2}{5}(\alpha+3\beta) + \frac{12}{35}(\alpha\beta+\beta^2) - \frac{3}{70}(3\alpha\beta^2+\beta^3) + \frac{2}{105} \alpha\beta^3 \right) \\
& - \kappa^2 a_1 \left(\frac{104}{45} - \frac{584}{315}(\alpha+\beta) + \frac{5353}{3465} \alpha\beta \right) \\
& + 20a_2 \left(\frac{26}{5} - \frac{32}{35}(\alpha+3\beta) + \frac{57}{70}(\alpha\beta+\beta^2) - \frac{11}{105}(3\alpha\beta^2+\beta^3) + \frac{1}{21} \alpha\beta^3 \right) \\
& - \kappa^2 a_2 \left(\frac{2644}{315} - \frac{6679}{990}(\alpha+\beta) + \frac{8669}{1540} \alpha\beta \right) \\
& + 30a_3 \left(\frac{284}{315} - \frac{103}{70}(\alpha+3\beta) + \frac{47}{35}(\alpha\beta+\beta^2) - \frac{37}{210}(3\alpha\beta^2+\beta^3) + \frac{94}{1155} \alpha\beta^3 \right) \\
& - \kappa^2 a_3 \left(\frac{45541}{2310} - \frac{439757}{27720} (\alpha+\beta) + \frac{238727}{18018} \alpha\beta \right) = 0, \quad (2.3.9)
\end{aligned}$$

$$\begin{aligned}
& 12a_1 \left(\frac{26}{3} - \frac{32}{21}(\alpha+3\beta) + \frac{19}{14}(\alpha\beta+\beta^2) - \frac{11}{63}(3\alpha\beta^2+\beta^3) + \frac{5}{63} \alpha\beta^3 \right) \\
& - \kappa^2 a_1 \left(\frac{2644}{315} - \frac{6679}{990}(\alpha+\beta) + \frac{8669}{1540} \alpha\beta \right) \\
& + 20a_2 \left(\frac{132}{7} - \frac{7}{2}(\alpha+3\beta) + \frac{68}{21}(\alpha\beta+\beta^2) - \frac{3}{7}(3\alpha\beta^2+\beta^3) + \frac{46}{231} \alpha\beta^3 \right) \\
& - \kappa^2 a_2 \left(\frac{21128}{693} - \frac{6181}{252}(\alpha+\beta) + \frac{184799}{9009} \alpha\beta \right) \\
& + 30a_3 \left(\frac{59}{2} - \frac{356}{63}(\alpha+3\beta) + \frac{75}{14}(\alpha\beta+\beta^2) - \frac{164}{231}(3\alpha\beta^2+\beta^3) + \frac{43}{126} \alpha\beta^3 \right) \\
& - \kappa^2 a_3 \left(\frac{12031}{168} - \frac{57797}{1001}(\alpha+\beta) + \frac{12167}{262} \alpha\beta \right) = 0, \quad (2.3.10)
\end{aligned}$$

$$\begin{aligned}
& 12a_1 \left(\frac{142}{7} - \frac{103}{28} (\alpha+3\beta) + \frac{47}{14}(\alpha\beta+\beta^2) - \frac{37}{84}(3\alpha\beta^2+\beta^3) + \frac{47}{231} \alpha\beta^3 \right) \\
& - \kappa^2 a_1 \left(\frac{45541}{2310} - \frac{439787}{27720}(\alpha+\beta) + \frac{238727}{18018} \alpha\beta \right)
\end{aligned}$$

$$\begin{aligned}
& + 20a_2 \left(\frac{177}{4} - \frac{178}{21}(\alpha+3\beta) + \frac{225}{28}(\alpha\beta+\beta^2) - \frac{167}{154}(3\alpha\beta^2+\beta^3) + \frac{43}{84} \alpha\beta^3 \right) \\
& \quad - \kappa^2 a_2 \left(\frac{12031}{168} - \frac{57797}{1001}(\alpha+\beta) + \frac{12167}{252} \alpha\beta \right) \\
& + 30a_3 \left(\frac{208}{3} - \frac{96}{7}(\alpha+3\beta) + \frac{1026}{77}(\alpha\beta+\beta^3) - \frac{11}{6}(3\alpha\beta^2+\beta^3) + \frac{80}{91} \alpha\beta^3 \right) \\
& \quad - \kappa^2 a_3 \left(\frac{15308}{91} - \frac{10456}{77}(\alpha+\beta) + \frac{393781}{3465} \alpha\beta \right) = 0. \quad (2.3.11)
\end{aligned}$$

The determinant of the coefficients of a_1 , a_2 and a_3 must be set equal to zero to get a non-trivial solution to Equations (2.3.9), (2.3.10), and (2.3.11). The coefficients of a_1 and a_2 in Equations (2.3.9) and (2.3.10) are identical to the coefficients of a_1 and a_2 in Equations (2.3.6) and (2.3.7). If the remaining coefficients are defined as

$$A_{13} = 30 \left(\frac{284}{315} - \frac{103}{70}(\alpha+3\beta) + \frac{47}{35}(\alpha\beta+\beta^2) - \frac{37}{210}(3\alpha\beta^2+\beta^3) + \frac{94}{1155} \alpha\beta^3 \right)$$

$$B_{13} = \frac{45541}{2310} - \frac{439757}{27720}(\alpha+\beta) + \frac{238727}{18018} \alpha\beta$$

$$A_{23} = 30 \left(\frac{59}{2} - \frac{356}{63}(\alpha+3\beta) + \frac{75}{14}(\alpha\beta+\beta^2) - \frac{164}{231}(3\alpha\beta^2+\beta^3) + \frac{43}{126} \right)$$

$$B_{23} = \frac{12031}{168} - \frac{57797}{1001}(\alpha+\beta) + \frac{12167}{252} \alpha\beta$$

$$A_{31} = 12 \left(\frac{142}{7} - \frac{103}{28}(\alpha+3\beta) + \frac{47}{14}(\alpha\beta+\beta^2) - \frac{37}{84}(3\alpha\beta^2+\beta^3) + \frac{47}{231} \alpha\beta^3 \right)$$

$$A_{32} = 20 \left(\frac{177}{4} - \frac{178}{21}(\alpha+3\beta) + \frac{225}{28}(\alpha\beta+\beta^2) - \frac{167}{154}(3\alpha\beta^2+\beta^3) + \frac{43}{84} \alpha\beta^3 \right)$$

$$A_{33} = 30\left(\frac{208}{3} - \frac{96}{7}(\alpha+3\beta) + \frac{1026}{77}(\alpha\beta+\beta^2) - \frac{11}{6}(3\alpha\beta^2+\beta^3) + \frac{80}{91}\alpha\beta^3\right)$$

$$B_{33} = \frac{15308}{91} - \frac{10456}{77}(\alpha+\beta) + \frac{393781}{3465}\alpha\beta,$$

Equations (2.3.9), (2.3.10), and (2.3.11) can be expressed as

$$\sum_{j=1}^3 (A_{ij} - \kappa^2 B_{ij}) a_j = 0, \quad i = 1, 2, 3.$$

Observing that $A_{13} = A_{31}$, and $A_{23} = A_{32}$ the determinant of the coefficients of a_1 , a_2 , and a_3 becomes

$$\begin{vmatrix} A_{11} - \kappa^2 B_{11} & A_{12} - \kappa^2 B_{12} & A_{13} - \kappa^2 B_{13} \\ A_{12} - \kappa^2 B_{12} & A_{22} - \kappa^2 B_{22} & A_{23} - \kappa^2 B_{23} \\ A_{13} - \kappa^2 B_{13} & A_{23} - \kappa^2 B_{23} & A_{33} - \kappa^2 B_{33} \end{vmatrix}.$$

In order to determine the natural frequencies, the determinant of the coefficients must be set equal to zero.

$$\text{i.e. } |A - \kappa^2 B| = 0.$$

When the determinant is expanded, it becomes necessary to find the differences between very large numbers. These differences are so small that the round off errors are large enough to produce erroneous results. To avoid this, the values of κ^2 are found in the following manner.

Both $[A]$ and $[B]$ are real symmetric matrices and the problem

$$|A - \kappa^2 B| = 0,$$

is equivalent to

$$\begin{bmatrix} A \end{bmatrix} \bar{V} = \kappa^2 \begin{bmatrix} B \end{bmatrix} \bar{V}, \quad (2.3.12)$$

where $\bar{V}$ is a column vector consisting of the elements a_1 , a_2 , and a_3 . Since $\begin{bmatrix} B \end{bmatrix}$ is a real symmetric matrix it may be expressed as the product

$$\begin{bmatrix} B \end{bmatrix} = \begin{bmatrix} L \end{bmatrix} \begin{bmatrix} L \end{bmatrix}^T,$$

where $\begin{bmatrix} L \end{bmatrix}$ is a lower triangular matrix.

Using this expression for $\begin{bmatrix} B \end{bmatrix}$ in Equation (2.3.12), gives

$$\begin{bmatrix} A \end{bmatrix} \bar{V} = \kappa^2 \begin{bmatrix} L \end{bmatrix} \begin{bmatrix} L \end{bmatrix}^T \bar{V},$$

which can be reduced to

$$\begin{bmatrix} L \end{bmatrix}^{-1} \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} L \end{bmatrix}^T^{-1} \bar{V} = \kappa^2 \bar{V},$$

and the problem of finding the values of κ^2 becomes the problem of finding the characteristic roots of the matrix

$$\begin{bmatrix} L \end{bmatrix}^{-1} \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} L \end{bmatrix}^T^{-1}.$$

This was done, without expanding the matrix, by using an iterative procedure [11].

2.3-4 Wedges With Shearing Force and Rotatory Inertia

The equation of motion for a wedge when the effects of shearing force and rotatory inertia are considered is

$$\begin{aligned}
& \frac{1}{c\ell^4} \frac{d^4 y}{dX^4} \left[E(1-\beta X)^2 \left(D - \rho\omega^2(1-\beta X)^2 + Ec^2 \right) \right] \\
& - \frac{1}{\ell^3} \frac{d^3 y}{dX^3} \left[E(1-\beta X) \left(6D - 4\rho\omega^2(1-\beta X)^3 + 6Ec^2 \right) \right] \\
& + \frac{1}{c\ell^2} \frac{d^2 y}{dX^2} \left[(D+Ec^2) \left(6Ec^2 + \rho\omega^2(1-\beta X)^2 \right) - \rho^2\omega^4(1-\beta X)^4 \right. \\
& \quad \left. + \frac{D+Ec^2}{k'G} E\rho\omega^2(1-\beta X)^2 - \frac{E\rho^2\omega^4}{k'G} (1-\beta X)^4 \right] \\
& - \frac{1}{\ell} \frac{dy}{dX} \left[(D+Ec^2) \left(3\rho\omega^2(1-\beta X) + \frac{5E\rho\omega^2}{k'G} (1-\beta X) \right) - \frac{3E\rho^2\omega^4}{k'G} (1-\beta X)^3 \right. \\
& \quad \left. - \rho^2\omega^4(1-\beta X)^3 \right] + y \left[\frac{(D+Ec^2)}{ck'G} \left(3Ec^2\rho\omega^2 - D\rho\omega^2 + 2\rho^2\omega^4(1-\beta X)^2 \right) \right. \\
& \quad \left. - \frac{\rho^3\omega^6}{ck'G} (1-\beta X)^4 \right] = 0. \tag{2.2.18}
\end{aligned}$$

When this equation is multiplied by c , and the coefficients of the powers of ω are collected it becomes

$$\begin{aligned}
& A_1 (1-\beta X)^4 y \omega^6 \\
& + \left[A_2 (1-\beta X)^4 y'' + A_3 (1-\beta X)^3 y' + A_4 (1-\beta X)^2 y \right] \omega^4 \\
& + \left[A_5 (1-\beta X)^4 y^{IV} + A_6 (1-\beta X)^3 y''' + A_7 (1-\beta X)^2 y'' + A_8 (1-\beta X) y' \right. \\
& \quad \left. + A_9 y \right] \omega^2 + \left[A_{10} (1-\beta X)^2 y^{IV} + A_{11} (1-\beta X) y''' + A_{12} y'' \right] = 0, \tag{2.3.13}
\end{aligned}$$

where primes denote differentiation with respect to X , and

$$A = D + Ec^2$$

$$A_7 = \frac{A \left(\rho + \frac{E\rho}{kG} \right)}{\ell^2}$$

$$A_1 = -\frac{\rho^3}{kG}$$

$$A_8 = \frac{-cA \left(3\rho + \frac{5E\rho}{kG} \right)}{\ell}$$

$$A_2 = -\left(\frac{\rho^2 + \frac{E\rho^2}{kG}}{\ell^2} \right)$$

$$A_9 = \frac{A}{kG} (3Ec^2\rho - D\rho)$$

$$A_3 = c \left(\frac{\frac{3E\rho^2}{kG} + \rho^2}{\ell} \right)$$

$$A_{10} = \frac{EA}{\ell^4}$$

$$A_4 = \frac{2\rho^2 A}{kG}$$

$$A_{11} = -\frac{6EAc}{\ell^3}$$

$$A_5 = \frac{-E\rho}{\ell^4}$$

$$A_{12} = \frac{6Ec^2 A}{\ell^2}$$

$$A_6 = \frac{4Ec\rho}{\ell^3}$$

When the one term trial solution for $y(x)$,

$$y_1(x) = a_1(6X^2 - 4X^3 + X^4), \quad (2.3.2)$$

is used in applying Galerkin's method to Equation (2.3.13), a cubic equation in ω^2 is produced. (See Appendix I-B),

2.3-5 Fundamental Frequency of Nonlinearly Tapered Beams

For beams with nonlinear tapers, where the moment of inertia and area can be expressed as

$$I = I_0(1-\alpha X^n)(1-\beta X^m)^3 \quad (2.2.10)$$

$$A = A_0 (1 - \alpha X^n) (1 - \beta X^m) , \quad (2.2.11)$$

the equation of motion is

$$\begin{aligned} (1 - \alpha X^n) (1 - \beta X^m)^3 y^{IV} - \frac{2}{\ell} \left[n \alpha X^{n-1} (1 - \beta X^m)^3 + 3m \beta X^{m-1} (1 - \alpha X^n) \right. \\ \left. (1 - \beta X^m)^2 \right] y''' + \frac{1}{\ell^2} \left[-n(n-1) \alpha X^{n-2} (1 - \beta X^m)^3 + 6mn \alpha \beta X^{m+n-2} \right. \\ \left. (1 - \beta X^m)^2 - 3m(m-1) \beta X^{m-2} (1 - \alpha X^n) (1 - \beta X^m)^2 + 6m^2 \beta^2 X^{2m-2} \right. \\ \left. (1 - \alpha X^n) (1 - \beta X^m) \right] y'' = \frac{\omega^2 \rho A_0}{EI_0} (1 - \alpha X^n) (1 - \beta X^m) y \quad (2.2.12) \end{aligned}$$

Using the one term trial solution ($y_1(x)$) in the application of Galerkin's method to this equation gives

$$\begin{aligned} & \int_0^1 L(y_1) \phi_1 dx \\ &= \int_0^1 \left[12a_1 \left[2(1 - \alpha X^n) (1 - \beta X^m)^3 + 12m \beta X^{m-1} (1 - \alpha X^n) (1 - \beta X^m)^2 (1 - X) \right. \right. \\ &+ 4n \alpha X^{n-1} (1 - \beta X^m)^3 (1 - X) + 6m^2 \beta^2 X^{2m-2} (1 - \alpha X^n) (1 - \beta X^m) (1 - 2X + X^2) \\ &- 3m(m-1) \beta X^{m-2} (1 - \alpha X^n) (1 - \beta X^m)^2 (1 - 2X + X^2) \\ &+ 6nm \alpha \beta X^{n+m-2} (1 - \beta X^m)^2 (1 - 2X + X^2) \\ &\left. \left. - n(n-1) \alpha X^{n-2} (1 - \beta X^m)^3 (1 - 2X + X^2) \right] \right. \\ &\left. - \frac{\rho A_0 \omega^2 a_1 \ell^4}{EI_0} (1 - \alpha X^n) (1 - \beta X^m) (6X^2 - 4X^3 + X^4) \right] (6X^2 - 4X^3 + X^4) dx = 0 , \end{aligned}$$

which can be solved for the fundamental frequency of vibration after the integration is completed.

2.4 Theoretical Results

The natural frequencies of vibration are expressed in the form ω/p , where ω is the natural frequency of vibration and p is the fundamental frequency of vibration of the corresponding prismatic beam, given by

$$p = \sqrt{\frac{EI_0}{\rho A_0}} \left(\frac{1.875}{l} \right)^2. \quad (2.2.5)$$

2.4-1 Convergence of the Results

To illustrate how the results obtained using Galerkin's method converge, the fundamental frequency of a linearly tapered beam was calculated using the one, two, and three term trial solutions, and the frequency of the second mode was calculated using the two and three term solutions. The values obtained for each mode were then compared to each other, and to the values obtained from the closed form solutions. Table 2-1 gives the values obtained for the fundamental frequency of vibration of linearly tapered beams when the one, two and three term trial solutions were used in the application of Galerkin's method to the equation of motion. These results are presented graphically in Figure 2-3. Table 2-2 and Figure 2-4 give the values obtained for the frequency of the second mode of vibration when the two and

TABLE 2-1 FUNDAMENTAL MODE FREQUENCY RATIOS

ALPHA=0.0											
BETA	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
ONE											
TERM	1.004	1.021	1.043	1.070	1.105	1.149	1.206	1.281	1.386	1.539	1.785
TWO											
TERMS	1.000	1.012	1.027	1.044	1.065	1.092	1.128	1.177	1.250	1.367	1.579
THREE											
TERMS	1.000	1.012	1.026	1.043	1.063	1.088	1.120	1.163	1.226	1.330	1.532

ALPHA=0.6											
BETA	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
ONE											
TERM	1.323	1.346	1.375	1.410	1.454	1.510	1.580	1.672	1.796	1.971	2.237
TWO											
TERMS	1.305	1.317	1.332	1.351	1.374	1.404	1.444	1.501	1.583	1.713	1.938
THREE											
TERMS	1.304	1.316	1.331	1.348	1.368	1.394	1.427	1.473	1.541	1.652	1.865

ALPHA=1.0											
BETA	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
ONE											
TERM	2.061	2.090	2.124	2.167	2.218	2.282	2.360	2.457	2.581	2.741	2.958
TWO											
TERMS	2.036	2.050	2.067	2.087	2.112	2.143	2.184	2.239	2.315	2.425	2.590
THREE											
TERMS	2.036	2.050	2.066	2.084	2.106	2.133	2.167	2.211	2.272	2.363	2.510

TABLE 2-2 SECOND MODE FREQUENCY RATIOS

ALPHA=0.0											
<u>BETA</u>	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
<u>TWO</u>											
<u>TERMS</u>	6.462	6.354	6.263	6.193	6.146	6.130	6.150	6.219	6.361	6.635	7.246
<u>THREE</u>											
<u>TERMS</u>	6.268	6.071	5.871	5.671	5.475	5.287	5.118	4.980	4.897	4.928	5.287

ALPHA=1.0											
<u>BETA</u>	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
<u>TWO</u>											
<u>TERMS</u>	9.327	9.215	9.125	9.062	9.029	9.029	9.069	9.172	9.335	9.617	10.114
<u>THREE</u>											
<u>TERMS</u>	8.831	8.575	8.318	8.065	7.820	7.590	7.384	7.215	7.107	7.100	7.295

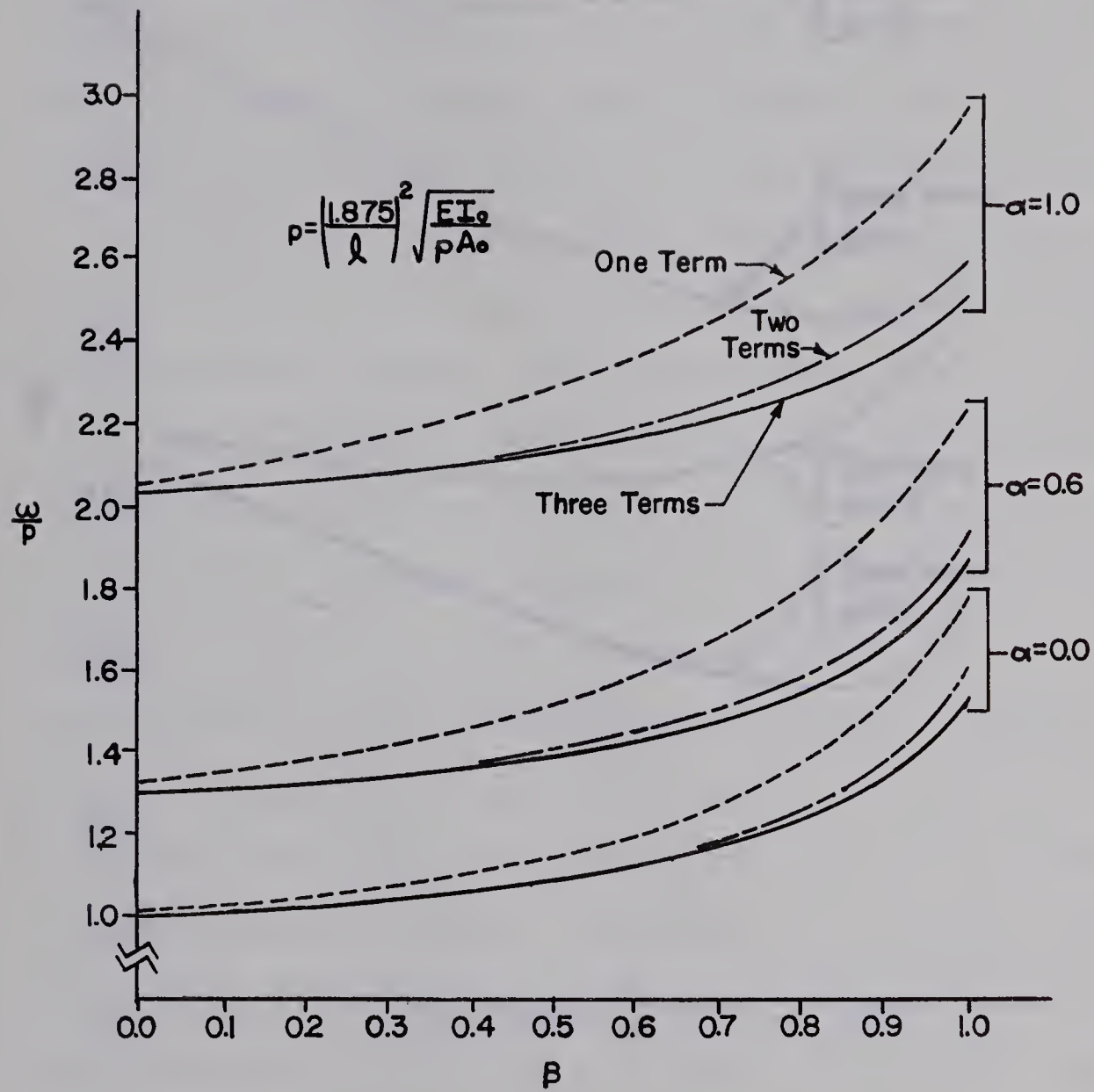


Figure 2-3 Convergence of Galerkin's Method to the Fundamental Frequency of Vibration

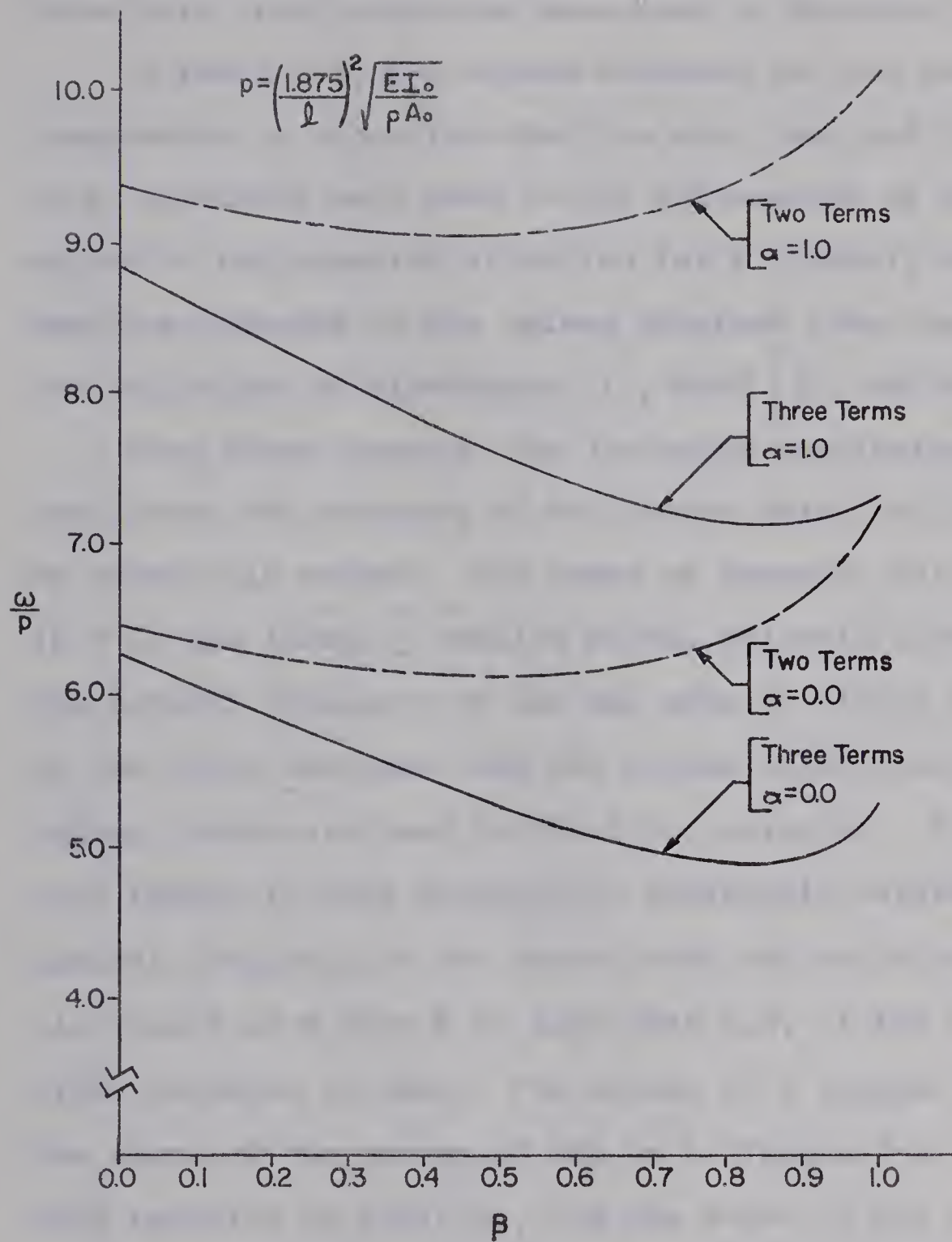


Figure 2-4 Convergence of Galerkin's Method to the Second Natural Frequency of Vibration

three term trial solutions were used in Galerkin's method.

In Table 2-3, the values obtained for the natural frequencies of vibration when the one, two, and three term trial solutions were used in the application of Galerkin's method to the equation of motion for a linearly tapered beam are compared to the values obtained from the closed form solutions of Timoshenko [1], Ward [2], and Wrinch [3].

From these results, the following conclusions can be made about the accuracy of the values which can be obtained by using this method. For beams of constant thickness ($\beta = 0$) and linearly varying width, Galerkin's method gives the natural frequency of the 2nd mode to within 0.1 per cent of the value obtained from the closed form solution when three terms are used in the trial solution. For beams with tapers in both directions, acceptable values for the natural frequency of the second mode can be calculated for all values of α when β is less than 0.8, if the three term trial solution is used. For values of β greater than 0.8 the slope of the curves of ω/p vs β (Figure 2-2) changes from negative to positive, and the error in the calculated value of the frequency increases.

2.4-2 Natural Frequencies of Vibration of Linearly Tapered Beams

The values of the frequencies of vibration of the

TABLE 2-3 NATURAL FREQUENCIES OF VIBRATION FOR THE LIMITING CASES

ALPHA \ BETA	0.0						1.0		
	MODE						MODE		
	SOLUTION	FUNDAMENTAL	SECOND	THIRD	FUNDAMENTAL	SECOND	THIRD	SECOND	THIRD
0.0	ONE TERM	1.004	*****	*****	1.785	*****	*****	*****	*****
	TWO TERMS	1.000	6.462	*****	1.579	7.246	*****	*****	*****
	THREE TERMS	1.000	6.280	18.846	1.532	5.287	19.111		
	MARTIN'S METHOD [8]	1.000	6.267	17.551	1.190	4.053	9.055		
	CLOSED FORM	1.000 [1]	6.267	17.551	1.512 [2]	4.326	8.553		
1.0	ONE TERM	2.016	*****	*****	2.958	*****	*****	*****	*****
	TWO TERMS	2.036	9.327	*****	2.590	10.114	*****	*****	*****
	THREE TERMS	2.036	8.831	23.923	2.510	7.295	24.516		
	MARTIN'S METHOD	1.498	7.178	18.456	1.688	4.764	9.868		
	CLOSED FORM	2.035*	8.830	21.465	2.480 [2,3]	6.016	10.955		

* The values for the case of $\alpha = 1.0$, $\beta = 0.0$ were taken from Martin's paper [8]. The credit for calculating these values belongs to F. Meyer.

first three modes were calculated using the three term trial solution. These results are presented in tabular form in Table 2-4, and graphically in Figures 2-5, 2-6, and 2-7. The broken sections of the curves in Figures 2-6 and 2-7 are for the values of β for which the calculated frequencies become less accurate.

From Figure 2-6 it can be observed that for any fixed value of α the second mode frequency varies as $1/\beta$, for values of β less than 0.5. If these linear sections are extrapolated to the values of $\beta = 1.0$, for the limiting cases of $\alpha = 0.0$ and $\alpha = 1.0$ the values obtained for ω/p are within five per cent of the values obtained from the closed form solutions. When a similar procedure is used to extrapolate the values of the third mode frequencies from $\beta = 0.2$, the frequencies obtained for $\beta = 1.0$ compare much better with the corresponding values obtained from the closed form solutions than the values obtained by using only Galerkin's method.

2.4-3 Natural Frequencies of Vibration When Shearing Force and Rotatory Inertia are Considered

When the effects of shearing force and rotatory inertia are considered, the common factor $\rho A_0 l^4 / EI_0$ cannot be taken out of the equation of motion, and hence the values of ω/p

TABLE 2-4 VALUES OF $\frac{\omega}{p}$ FOR THE FIRST THREE MODES OF VIBRATION OF

LINEARLY TAPERED CANTILEVER BEAMS

BETA ALPHA	MODE	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	1st	1.000	1.012	1.026	1.043	1.063	1.088	1.120	1.163	1.226	1.330	1.532
	2nd	6.268	6.071	5.871	5.671	5.475	5.287	5.118	4.980	4.897	4.928	5.287
	3rd	18.846	18.452	18.110	17.825	17.605	17.457	17.394	17.435	17.615	18.029	19.111
0.2	1st	1.070	1.082	1.097	1.113	1.133	1.158	1.191	1.235	1.299	1.405	1.612
	2nd	6.401	6.200	5.998	5.797	5.600	5.414	5.247	5.113	5.037	5.076	5.444
	3rd	19.178	18.807	18.488	18.227	18.029	17.903	17.862	17.923	18.125	18.563	19.673
0.4	1st	1.165	1.178	1.192	1.208	1.229	1.254	1.286	1.331	1.398	1.507	1.717
	2nd	6.577	6.373	6.168	5.964	5.768	5.583	5.419	5.290	5.221	5.270	5.648
	3rd	19.606	19.261	18.967	18.730	18.556	18.455	18.436	18.521	18.746	19.212	20.351
0.6	1st	1.304	1.316	1.331	1.348	1.368	1.394	1.427	1.473	1.541	1.652	1.865
	2nd	6.836	6.627	6.418	6.212	6.014	5.830	5.670	5.546	5.486	5.545	5.930
	3rd	20.199	19.880	19.613	19.402	19.254	19.176	19.181	19.291	19.543	20.039	21.207
0.8	1st	1.535	1.548	1.562	1.580	1.601	1.627	1.661	1.708	1.777	1.889	2.093
	2nd	7.307	7.088	6.870	6.658	6.455	6.268	6.106	5.985	5.929	5.996	6.371
	3rd	21.155	20.859	20.613	20.424	20.297	20.242	20.271	20.405	20.686	21.215	22.391
1.0	1st	2.036	2.050	2.066	2.084	2.106	2.133	2.167	2.211	2.272	2.363	2.510
	2nd	8.831	8.575	8.318	8.065	7.820	7.590	7.384	7.215	7.107	7.100	7.295
	3rd	23.923	23.570	23.272	23.036	22.869	22.779	22.778	22.886	23.138	23.610	24.576

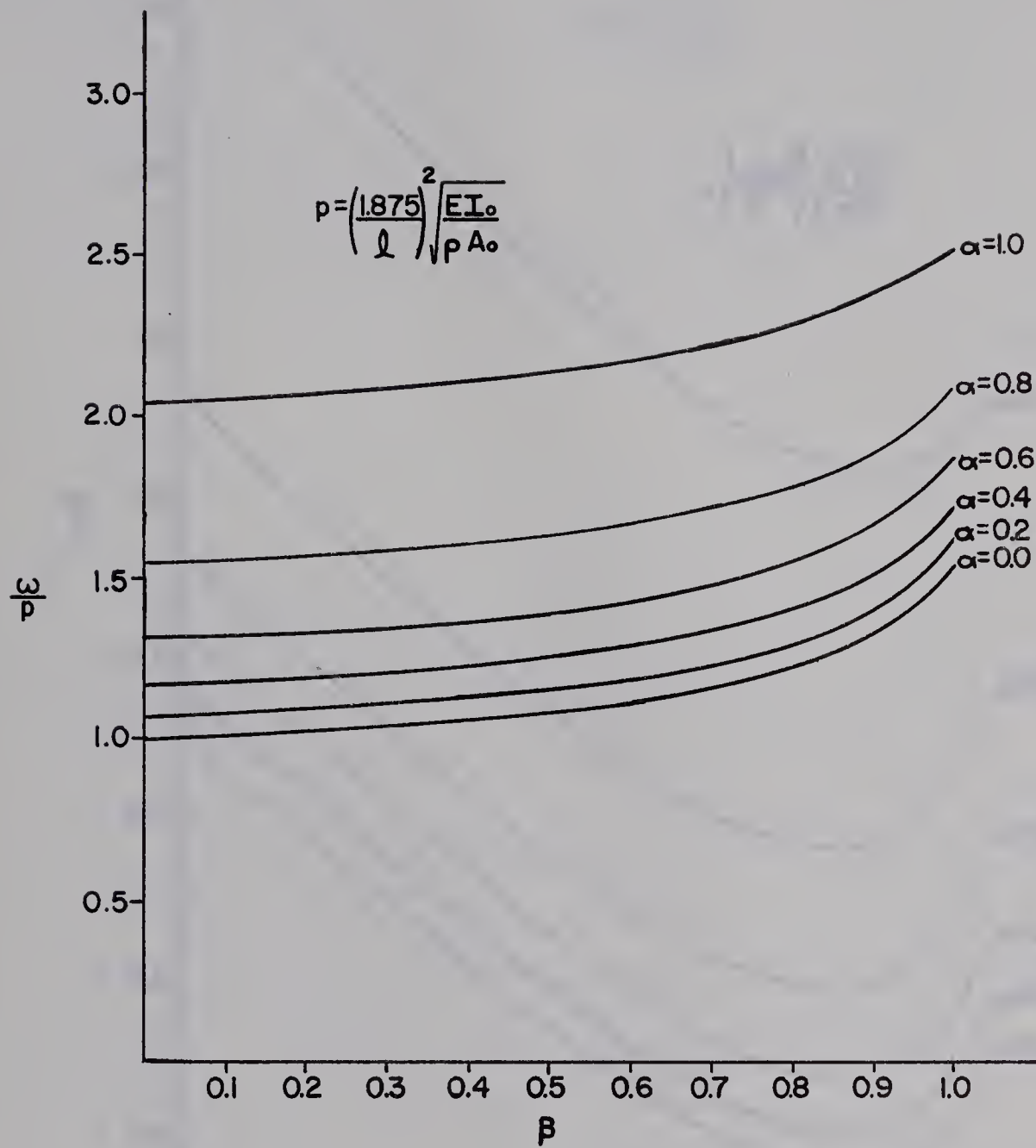


Figure 2-5 Fundamental Frequency of Vibration Versus Thickness Taper

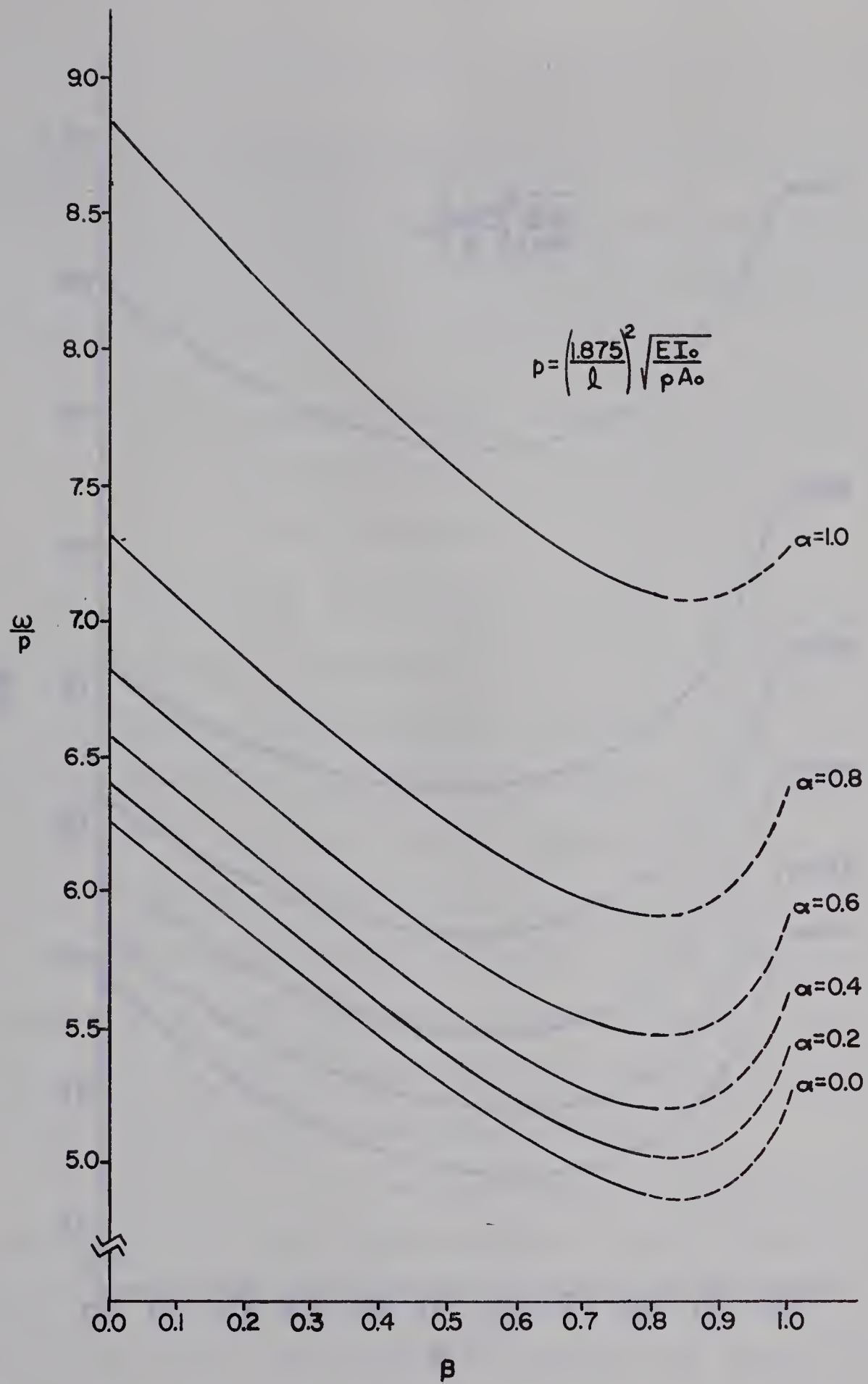


Figure 2-6 Second Natural Frequency of Vibration Versus Thickness Taper

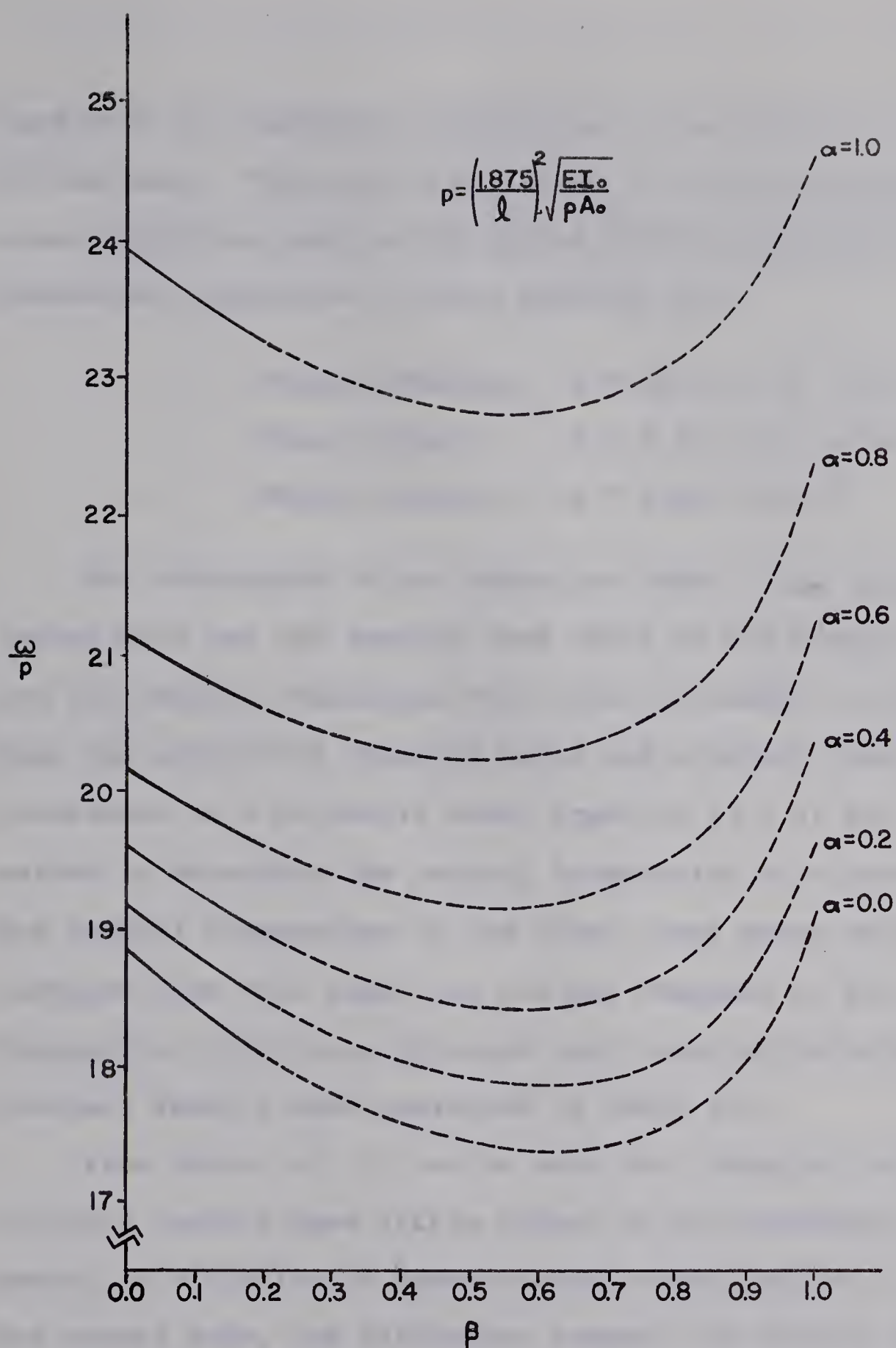


Figure 2-7 Third Natural Frequency of Vibration Versus Thickness Taper

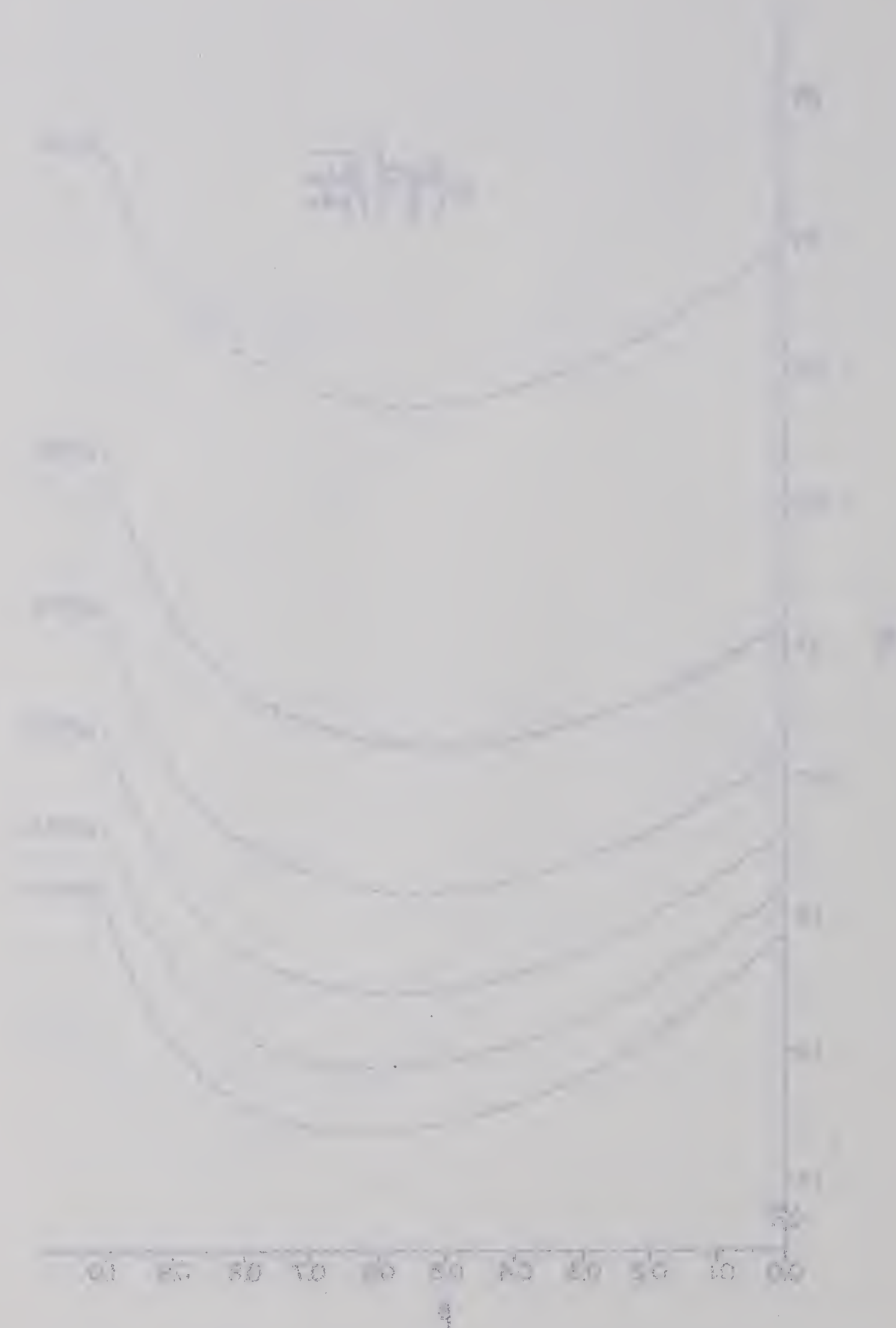


Figure 3. Third natural frequency of vibration versus thickness of plate.

vary with the mechanical properties of the material used in the beam. The results presented in this section are for beams which are made out of Alclad 7075-T6 aluminum*. The mechanical properties of this material are:

$$\text{Young's Modulus} \quad E = 10.4 \times 10^6 \text{ lb/in}^2.$$

$$\text{Shear Modulus} \quad G = 3.9 \times 10^6 \text{ lb/in}^2.$$

$$\text{Weight Density} \quad \rho g = 0.101 \text{ lb/in}^3.$$

The dimensions of the beams are taken to be three inches wide and one quarter inch thick at the clamped end, and the lengths considered vary from two inches to 16 inches. When the effects of shearing force and rotatory inertia are considered in a prismatic beam, Equation (2.2.9) has to be solved to determine the natural frequencies of vibration. The natural frequencies of the first three modes were determined from this equation, and are compared to the natural frequencies which were obtained when shearing force and rotatory inertia were neglected in Table 2-5.

From Table 2-5, it can be seen that shearing force and rotatory inertia have little effect on the fundamental frequency of vibration of beams of this cross section. For the second mode, the difference between the natural frequencies of vibration which were calculated when shearing force and

* This is the material which was used to make the test specimens.

TABLE 2-5 FREQUENCIES OF VIBRATION OF UNIFORM BEAMS

LENGTH INCHES	FREQUENCY CPS.					
	FIRST MODE		SECOND MODE		THIRD MODE	
	SIMPLE THEORY	SH.F.AND ROT.INR.	SIMPLE THEORY	SH.F.AND ROT.INR.	SIMPLE THEORY	SH.F.AND ROT.INR.
16	31.45	31.45	197.1	197.0	551.9	550.8
14	41.08	41.08	257.4	257.3	720.9	719.0
12	55.91	55.91	350.4	350.0	981.2	977.8
10	80.51	80.51	504.7	503.8	1412.9	1405.8
8	125.8	125.8	788.4	786.6	2207.7	2190.5
6	223.6	223.6	1402	1398	3925	3871
4	503.2	503.2	3154	3127	8831	8570
2	2012	2012	12608	12189	35326	31786

rotatory inertia were and were not considered varies as approximately $1/l$, to a maximum difference of -3.3 per cent of the frequency given by the simpler theory, for a two inch beam. Shearing force and rotatory inertia have a more pronounced effect on the natural frequency of the third mode. These effects reduce the frequency from 0.2 per cent for a 16 inch beam to 10 per cent for a two inch beam.

The fundamental frequency of vibration of wedges of lengths between two and 16 inches and tapers varying from $\beta = 0.0$ to $\beta = 1.0$ was calculated by applying the one term trial solution to Equation (2.3.13) (See Appendix I-B). The results which were obtained are presented in the form of frequency ratios (ω/p) in Figure 2-8, and Table 2-6. The frequency ratios which were obtained for the wedges between eight and 16 inches long were all equal (for equal tapers), and lower than the corresponding frequency ratios obtained when the three term trial solution was applied to the simpler equation where shearing force and rotatory inertia are neglected. These frequency ratios are given by the solid line in Figure 2-8. For wedges between eight and two inches long, the frequency ratios varied slightly, being higher than the frequency ratios for the longer wedges in the regions near $\beta = 0.0$ and $\beta = 1.0$, and lower than the frequency ratios for the longer wedges for values of β between

0.06 and 0.94. The frequency ratios for the two inch wedges are given by the broken line in Figure 2-8, and the frequency ratios for the wedges between two and eight inches long are all in the shaded area between the solid and broken lines.

The frequency ratios which were obtained, for the wedges between eight and 16 inches long, when the effects of shearing force and rotatory inertia were considered are compared to the values obtained from the simpler case when shearing force and rotatory inertia were neglected in Table 2-6.

TABLE 2-6 FUNDAMENTAL FREQUENCY RATIO FOR WEDGES
BETWEEN EIGHT AND 16 INCHES LONG

BETA	0.0	0.2	0.4	0.6	0.8	1.0
ONE TERM	1.004	1.043	1.105	1.206	1.386	1.785
THREE TERMS	1.000	1.026	1.063	1.120	1.226	1.532
SH. F. R.I.	1.004	1.026	1.045	1.064	1.080	1.096

From the values given in Table 2-6, it can be seen that the effects of shearing force and rotatory inertia partially counteract the effects of taper on the fundamental frequency of vibration.

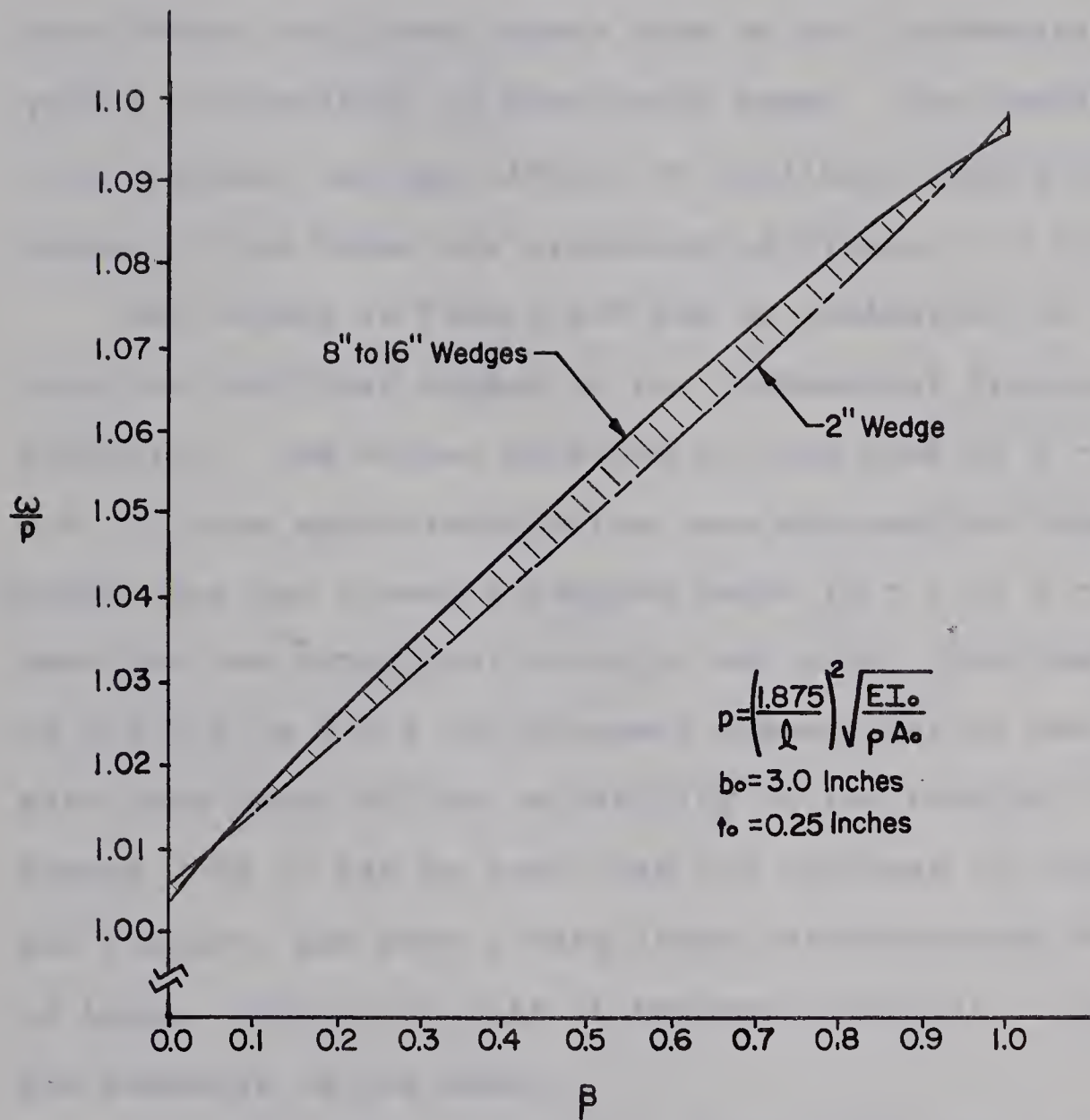


Figure 2-8 Fundamental Frequency of Vibration of Aluminum Wedges When Shearing Force and Rotatory Inertia are Considered

2.4-4 Effects of Nonlinear Tapers

The one term trial solution was used to determine what effect nonlinear tapers have on the fundamental frequency of vibration of cantilever beams. The results of this problem, and the effects of nonlinear tapers on the shapes of the beams are presented in Figures 2-9 and 2-10.

The curves in Figure 2-9 are an indication of the effects of nonlinear tapers on the fundamental frequency of vibration. The values obtained for the case of $n = 2.0$, $m = 2.0$ have approximately the same accuracy as those obtained for the linearly tapered beams ($n = 1.0$, $m = 1.0$), when the one term trial solution was used. For the case of $n = 0.5$, $m = 0.5$ the abnormal appearances of the curves cast some doubt on the reliability of the results. From Figure 2-10 it can be seen that the surfaces of this beam are concave, and have a very large curvature near the root of beam. Because of this it becomes difficult to predict the behavior of the beam.

For the values of n and m greater than one, the beam approaches the shape of a uniform beam, and the trial solution is a more accurate representation of the actual deflected form. This should give more accurate results. It can also be expected that the beam will behave more like a uniform beam in the higher modes of vibration.

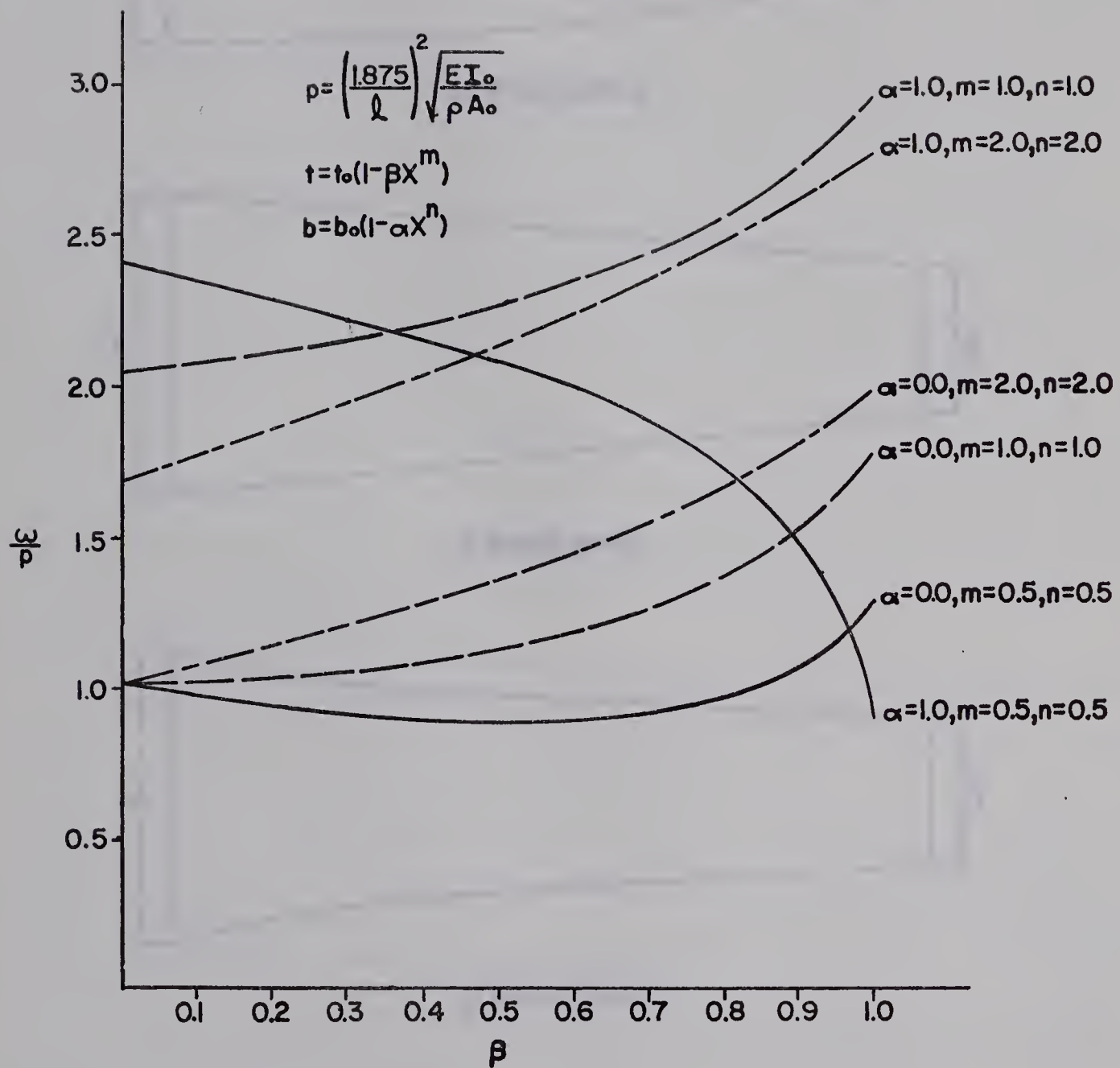


Figure 2-9 Fundamental Frequency of Vibration Versus Thickness Taper for Nonlinearly Tapered Beams

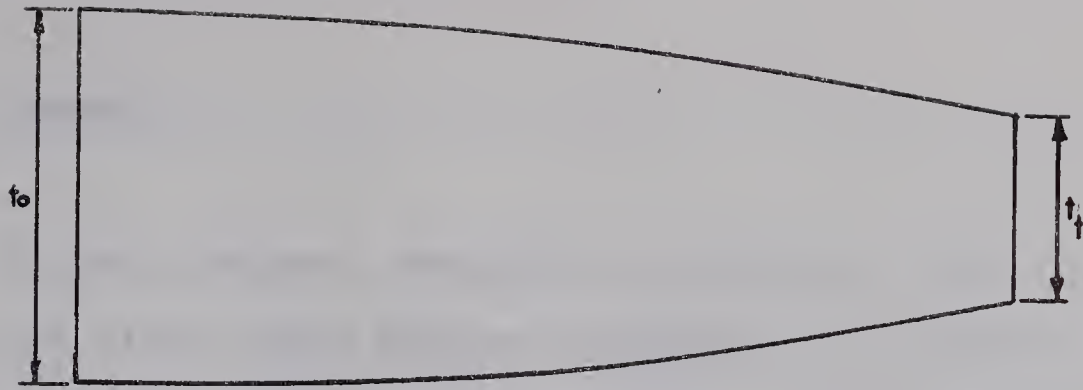
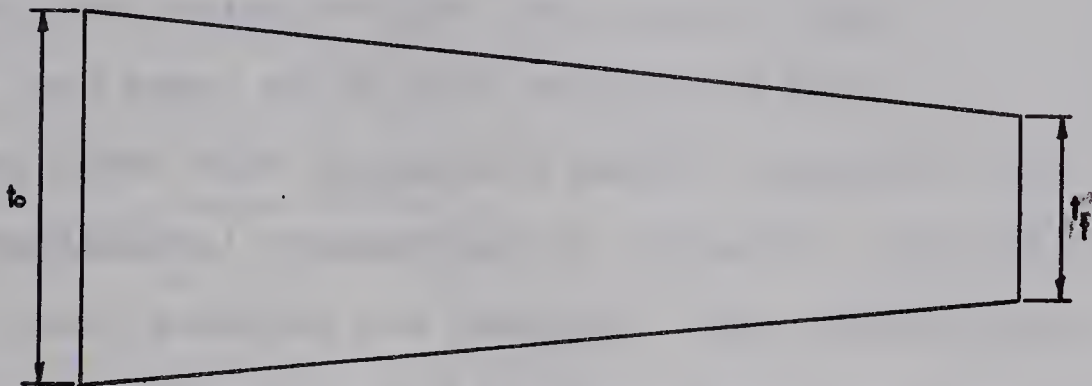
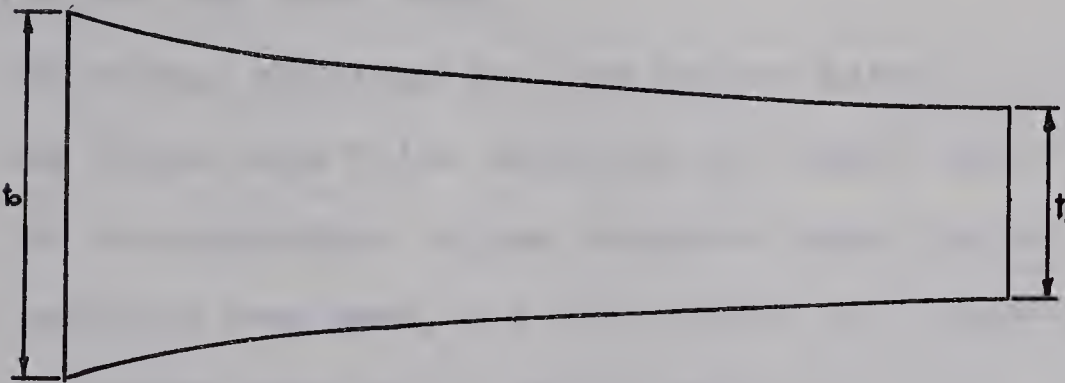

a) $\beta=0.5, m=2.0$

b) $\beta=0.5, m=1.0$

c) $\beta=0.5, m=0.5$

Figure 2-10 Beam Profiles

2.5 Summary

In this Chapter, Galerkin's method was used to calculate the first three natural frequencies of lateral vibration of linearly tapered cantilever beams. The frequencies obtained were compared to the frequencies obtained from the closed form solutions for the special cases of a uniform beam, and beams which have an area of zero at the free end. It was found that Galerkin's method converged very fast to the fundamental frequencies of vibration, and when the three term trial solution was used, the value calculated for the fundamental frequency of a pointed beam was less than 1.25 per cent higher than the frequency given by the closed form solution for the same case.

The values obtained for the second natural frequencies, when the three term trial solution was used, were much better than the corresponding values obtained when the two term trial solution was used, but for values of β greater than 0.8, the errors were still quite large, i.e. 30 - 40 percent.

Since at least n terms are required in the trial solution in order to compute a value for the n^{th} natural frequency, only a first approximation was obtained for the third natural frequencies, and the frequencies which were calculated were much too high to be of any practical use. Better values could be obtained for the third natural

frequency if more than three terms were used in the trial solution.

When the three term trial solution was used, the determinant of a three by three symmetric matrix had to be set equal to zero to determine the natural frequencies of vibration. When the determinant was expanded, it became necessary to find the difference between large numbers, and the round off errors which were introduced were sufficient to give erroneous results. To avoid this, the matrix was changed by means of a series of matrix multiplications and the frequencies were found by using an iterative procedure.

Also considered in this Chapter were the problems of finding the natural frequencies of vibration of linearly tapered cantilever beams when shearing force and rotatory inertia were considered, and the natural frequencies of vibration on nonlinearly tapered beams.

For the case when shearing force and rotatory inertia were considered, it was found that the two equations which govern the motion could be reduced to one equation for beams with constant width and linearly varying thickness. When Galerkin's method was applied to this equation using the one term trial solution, the values obtained for the fundamental frequency were much lower than the values obtained when the effects of shearing force and rotatory inertia were neglected. In view of this fact the results have to

be assumed to be inaccurate, however, the results did indicate that tapering the beams amplify the effects of shearing force and rotatory inertia.

Some experiments were conducted with linearly tapered cantilever beams to determine experimental values for the first three natural frequencies of vibration. In Chapter IV these experiments are discussed and the experimental results are presented and compared to the theoretical results.

CHAPTER III

EXPERIMENTS AND EXPERIMENTAL RESULTS

Frequency response spectrums were obtained for eight and 16 inch long aluminum beams by mounting each beam on an electrodynamic shaker table and recording the amplitudes of the deflections of the tip and root of the beam at various frequencies.

3.1 Apparatus

The vibrations in the beams were excited by an Unholtz-Dickie electrodynamic shaker, which was controlled by an automatic vibration exciter control. With this system, the frequency of the vibration and the amplitude of the support deflection could be set at any desired level which was within the limits of the testing apparatus. A frequency meter was connected to automatic vibration exciter control and the frequency of the vibration was read from it. The amplitude of the support deflection was read directly from the meter of the vibration exciter.

To measure the tip deflection, a stroboscope and a Gaertner cathetometer were used. The stroboscope had to be synchronized with the frequency of the shaker and the phase angle between the input signal and the output of the stroboscope had to be varied through 360 degrees in order

to determine the extreme positions of the tip of the beam. The frequency synchronizing and phase shifting was done automatically with a Chadwick-Helmuth Slip-Sync system. A signal from the vibration exciter was fed to the Slip-Sync system, which produced a pulse output at a frequency proportional to the input frequency and out of phase with the input signal by an angle which could be varied from zero to 360 degrees. The output from the Slip-Sync system was used to control the stroboscope. The Slip-Sync system was set to vary the phase angle continuously through 360 degrees, and when the tip of the beam was observed in the light from the stroboscope, it appeared to be oscillating slowly, and the amplitudes of the oscillations were measured with the cathetometer.

3.2 Test Specimens

The beams were machined from a sheet of Alclad 7075-T6 aluminum. The dimensions of the root were three inches wide by one quarter inch thick, and the lengths were eight inches and 16 inches. To maintain the boundary conditions at the clamped end of the beam, ($y(0) = 0$ and $\frac{dy(0)}{dx} = 0$) the beams were made symmetric about the clamped area. This produced two beams of identical geometry (See Figure 3-1).

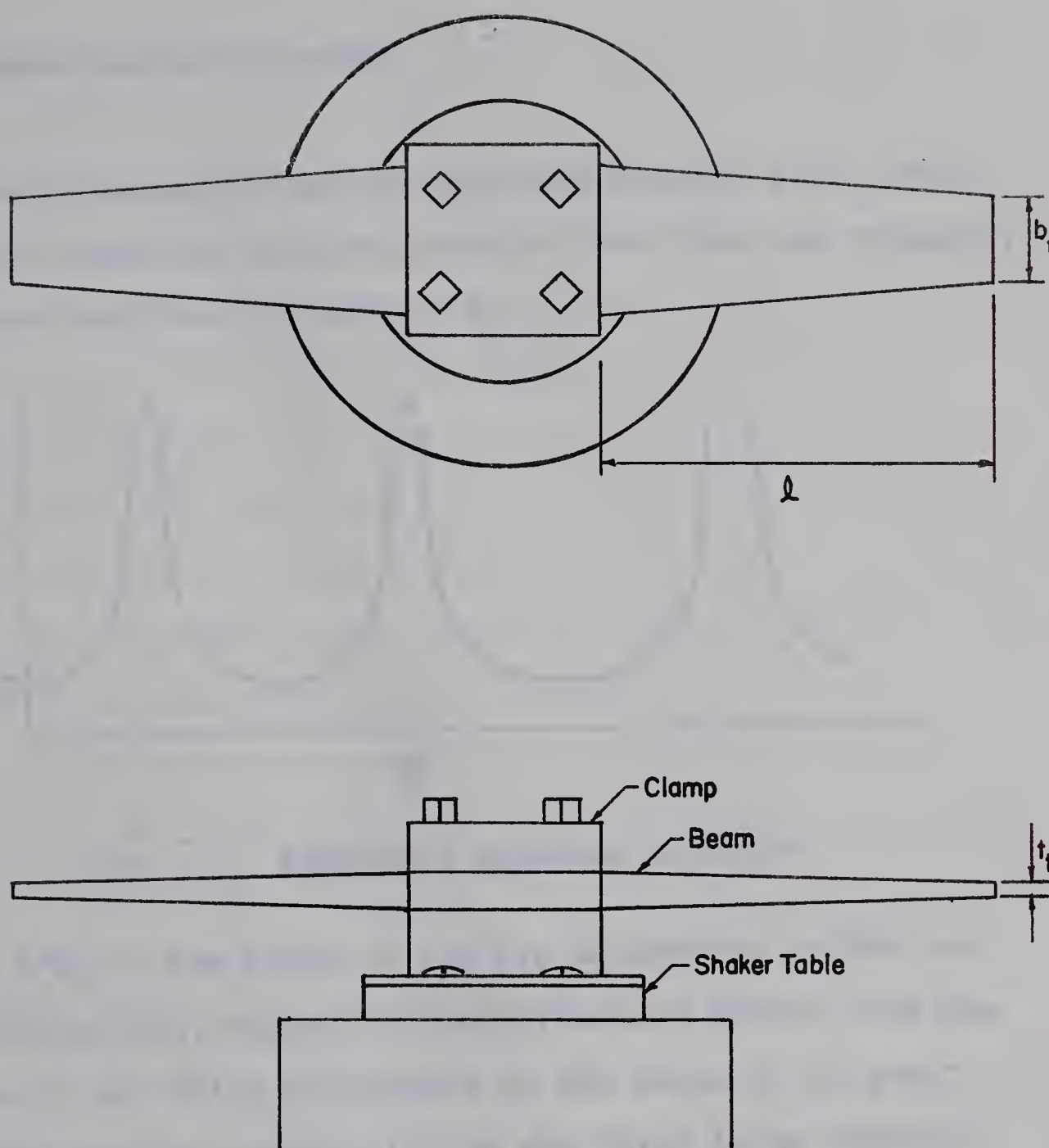


Figure 3-1 Specimen Configuration and Mounting

3.3 Experimental Procedure

The frequency response spectrums for the first three modes of vibration were expected to look like the frequency response spectrum in Figure 3-2,

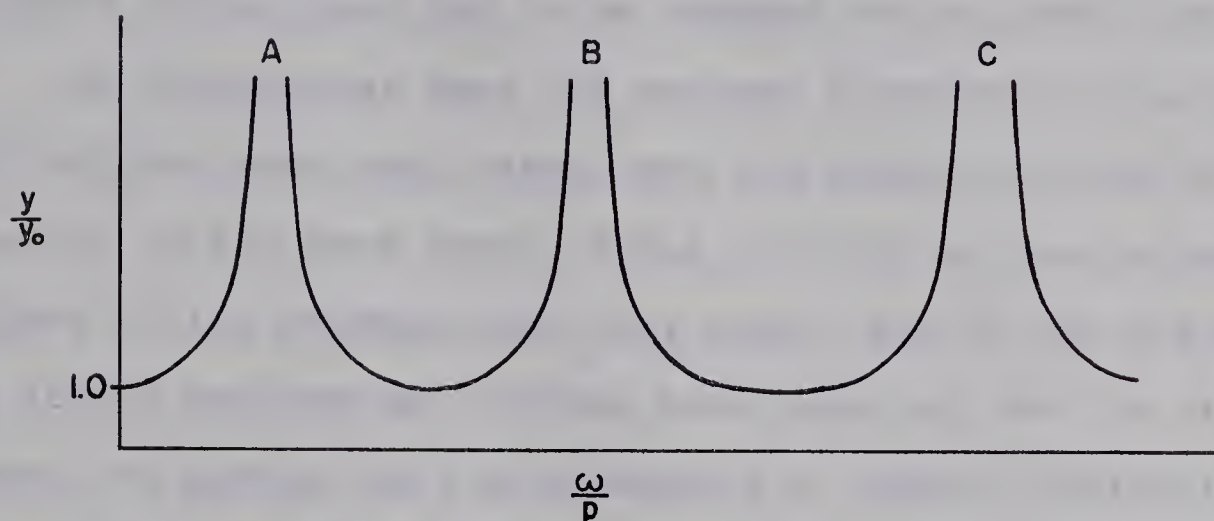


FIG. 3-2 FREQUENCY RESPONSE SPECTRUM

where y/y_0 is the ratio of the tip deflection to the support deflection, called the magnification factor, and the values of ω/p which correspond to the peaks A, B, and C give the natural frequencies of the first three modes of vibration.

The first step of each test was to determine the location of the first three peaks. The frequency was increased slowly from zero cycles per second to the frequency at which the third peak occurred, and the frequencies at which the peaks occurred were recorded. Following this step, the tip and root deflections were measured at various

frequencies between 15 cps and 1100 cps for the eight inch beams, and 15 cps and 500 cps for the 16 inch beams. It was initially planned to measure the tip deflections at all frequencies for a constant support deflection of 0.02 inches, but some difficulties were encountered, and the support deflections had to be changed during each test.

At frequencies near the natural frequencies the tip deflections were very large, and the amount of time that the tip of the beam spent in the vicinity of the extreme points of its movement was very small, and it was difficult to locate maximum and minimum positions for the tip of the beam. To reduce the tip movement the support deflection was reduced to 0.001 inches, and more accurate measurements of the tip movement could be made. When using this very small support deflection it became difficult to measure the tip deflection at frequencies which were not near the natural frequencies. At these frequencies, and with a support deflection of 0.001 inches, the tip of the beam was moving less than 0.004 inches. This small movement was hard to detect and could not be measured accurately, therefore larger support deflections had to be used for frequencies which were in this range.

Another problem arose from the fact that the force required to vibrate the beam varies as the product of the support deflection and the square of the frequency. Because

of this fact, the force required to vibrate the beam at high frequencies was greater than the shaker could produce. To expand the range of frequencies over which readings could be taken, the support deflections were decreased as the frequency was increased, until the tip movements became too small to be measured with the existing apparatus. As a result, the frequency spectrums could not be completed for the third modes of the eight inch beams. However, the natural frequencies of the third modes could still be determined because the tip movements became large enough to be seen when the third natural frequency was reached.

When it became necessary to vary the support deflections, a series of tests were conducted to determine whether or not the magnification factors varied linearly with the support deflections. These tests and the results obtained from them are discussed in Subsection 3.3-1.

Since the support deflections had to be varied, the tests were conducted in the following manner. The initial frequency for each test was 15 cps, and the frequency was increased by amounts varying from two cps for frequencies which were near the natural frequencies to 50 cps for frequencies which were far from the natural frequencies (i.e. in the regions where the slope of the frequency spectrum was near zero). The support and tip deflections were measured at each frequency when the support deflection was such that

the tip deflection was in the range where it could be measured most accurately (0.01 to 0.4 inches). The frequency spectrums obtained in this manner are presented in Section 3.4.

The values of the natural frequencies obtained from these tests were always lower than the theoretical values (See Table 3-2), and when the beams were examined after each test, abraded areas were found in the clamped area of the beams. This indicated that there was relative movement between the clamp and the beam, which could occur if the clamp was loose or deforming with the beams. Tests were then conducted to determine the effect of clamping pressure on the fundamental frequency of vibration of the beam (See Subsection 3.3-2). It was found that by changing the clamping pressure, no appreciable change in the fundamental frequency could be induced. The possibility of the clamp deforming with the beam is discussed in Section 3-4.

3.3-1 Linearity of the System

In order to maintain tip deflections which could be measured accurately, it was necessary to vary the support deflections as the frequency was changed. A series of tests was conducted to determine whether or not the magnification factor ($\frac{\text{tip deflection}}{\text{root deflection}}$) remained constant as the support

deflection was varied through the range of values for which the tip deflections could be measured at a constant frequency. A typical set of results is presented in Figure 3-3. From Figure 3-3, it can be seen that at frequencies which are not near a natural frequency, the magnification factor (y/y_0) decreases slightly as the support deflections are increased. For the cases when the frequencies were near the natural frequencies of vibration (in this case 30 cps for the first mode and 180 cps for the second mode) the magnification factors were large and unstable. In these regions the tip deflections would vary considerably when the support deflection and frequency were left constant, and the tip deflections could not be measured accurately.

3.3-2 Effects of Clamping Pressure on the Fundamental Frequency

The abrasions found on the clamped area of the beams, indicated that the beams were moving in the clamp. If the movement between the clamp and the beam could be reduced by increasing the clamp pressure, the natural frequency of vibration of the beam would increase because the effective length of the beam would be reduced. A series of experiments was conducted to determine what effect the clamp pressure had on the natural frequency of vibration.

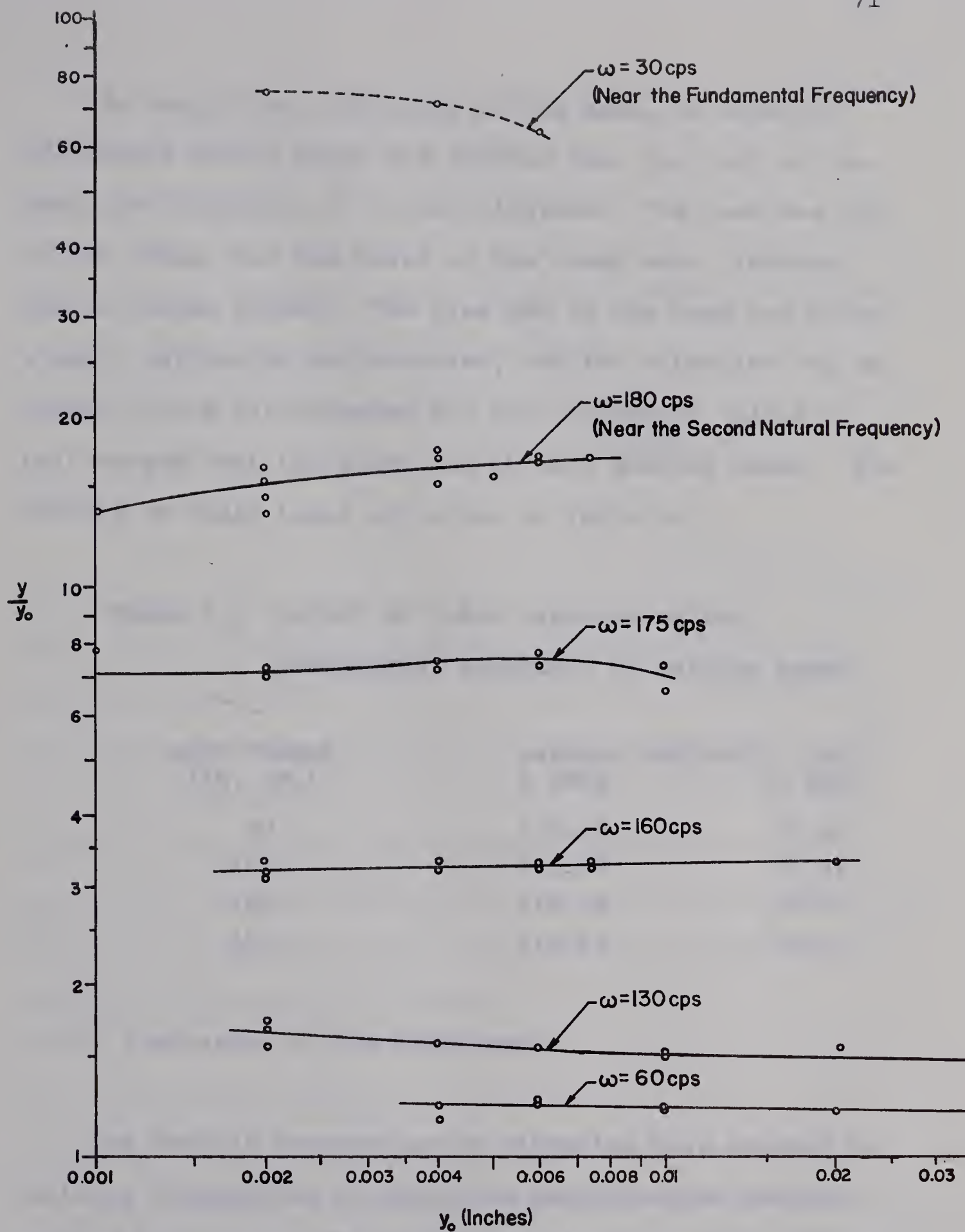


Figure 3-3 Magnification Factor Versus Support Deflection for a 16 Inch Uniform Beam

To record the vibrations of the beam, an electric resistance strain gauge was mounted near the root of the beam, and connected to an oscillograph. The beam was put in the clamp, and the bolts on the clamp were tightened with a torque wrench. The free end of the beam was given a small deflection and released, and the vibration was recorded. This was repeated for four different values of bolt torque with the eight and 16 inch uniform beams. The results of these tests are given in Table 3-1.

TABLE 3-1 EFFECT OF CLAMP PRESSURE ON THE
FUNDAMENTAL FREQUENCY OF UNIFORM BEAMS

BOLT TORQUE (IN. LB.)	NATURAL FREQUENCY (CPS)	
	8 INCH	16 INCH
80	116.16	30.14
100	116.30	30.16
125	116.54	30.17
150	116.67	30.17

3.3-3 Discussion of the Experiments

The natural frequencies of vibration were assumed to be those frequencies at which the magnification factors reached a maximum. This assumption is correct when there is no damping, but when damping is present, the frequency

at which the maximum magnification occurs is $\omega/p = \sqrt{1 - 2\delta^2}$, and the natural frequency is $\omega/p = \sqrt{1 - \delta^2}$, where δ is the damping factor. From the expressions for ω/p , it can be seen that for small values of δ the natural frequency is very close to the frequency at which the maximum magnification occurs. Since the damping was assumed to be small for the beams being tested, the assumption that the natural frequency is the frequency at which the maximum magnification occurs is valid.

In view of the results of the tests conducted for Subsection 3.3-1, it must be kept in mind that when viewing the frequency spectrums the values of y/y_0 greater than seven are possibly in error, because of the instability of the system when the magnification factors are large. However, this does not have a serious effect on the results which are desired from the frequency spectrums, since the main object of these experiments was to determine the natural frequency of vibration, and not the maximum magnification factors which could be obtained.

3.4 Experimental Results

The frequency spectrums presented in this section were obtained by measuring the amplitudes of the tip deflections (y) when the vibrations were excited by a known support

deflection (y_0). The magnification factors are defined as y/y_0 and the frequencies are given in the form ω/p , where ω is the exciting frequency and p is the fundamental frequency of the corresponding uniform beam. The frequency spectrums obtained in this manner are presented in Figures 3-4 to 3-10, and the experimental values of the natural frequencies are given in Table 3-2.

3.4-1 Discussion of the Experimental Results

On the basis of the experimental results which were obtained and observations which were made while the experiments were being conducted, the following conclusions were made about the experimental results.

The natural frequencies obtained experimentally were lower than the values given by Galerkin's method, and also generally lower than the values which were computed using Martin's method. It is also apparent that the longer beams gave better experimental values than the shorter beams. On examining each beam after the tests were completed, an abraded area was found near the edge of the clamp. This indicated that there was relative movement between the surfaces of the clamp and the surfaces of the beam. Since the clamp pressure did not affect the natural frequency of vibration (See Subsection 3.3-2), the clamp had to be deforming

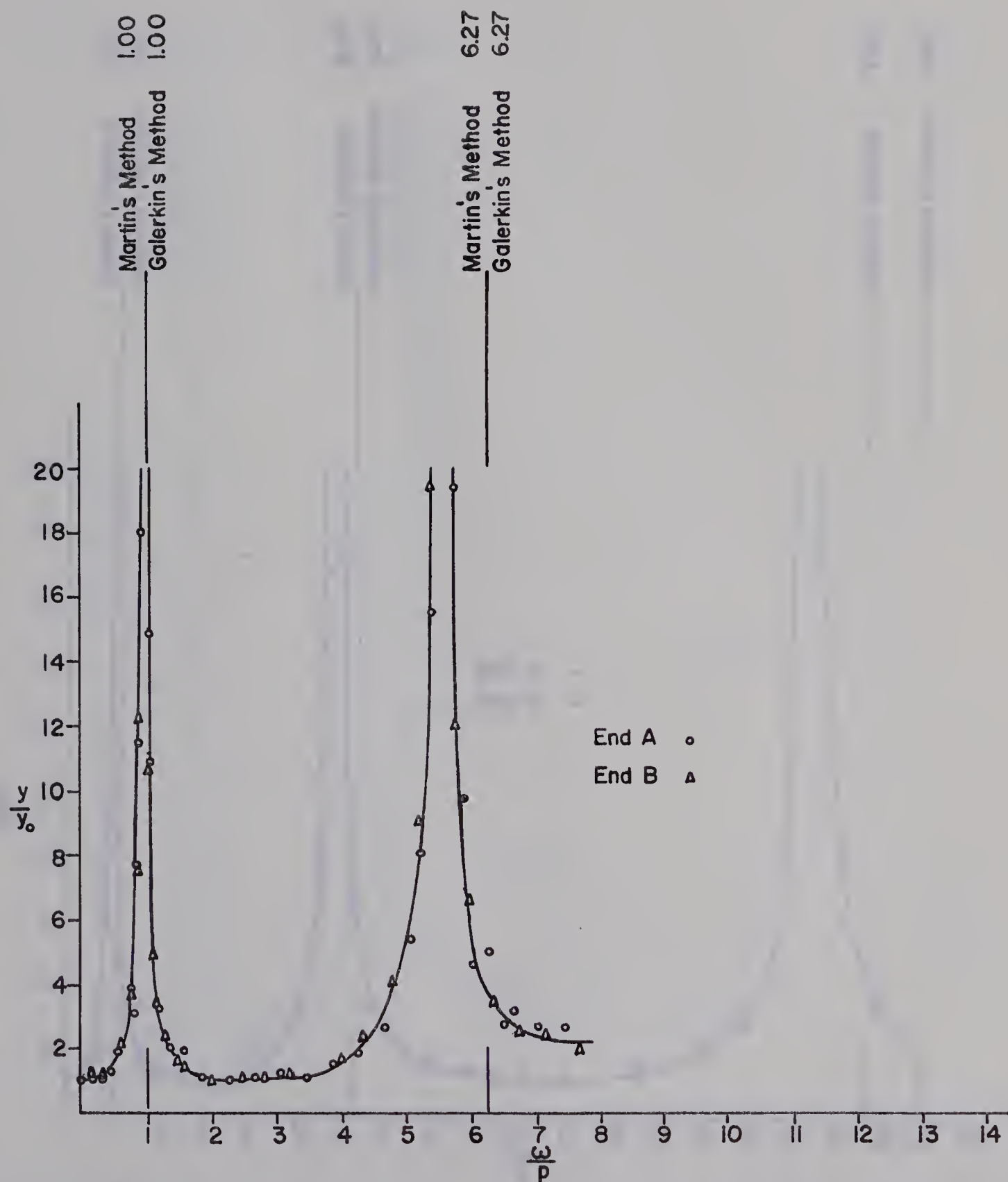


Figure 3-4 Frequency Spectrum for an 8" Aluminum Beam $\alpha=0.0$, $\beta=0.0$, $p=125.9$ cps

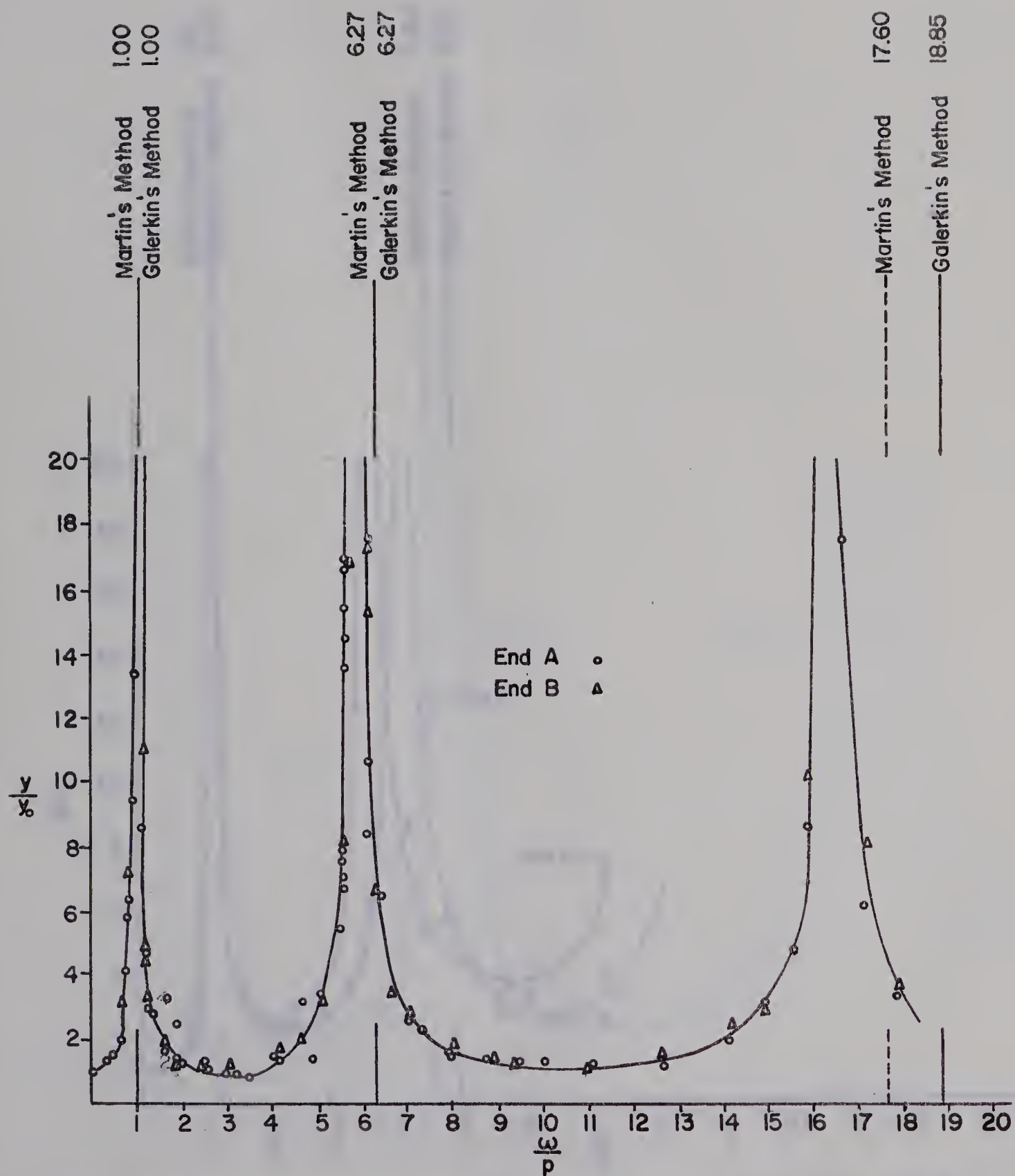


Figure 3-5 Frequency Spectrum for a 16" Aluminum Beam $\alpha=0.0$, $\beta=0.0$, $p=31.5$ cps

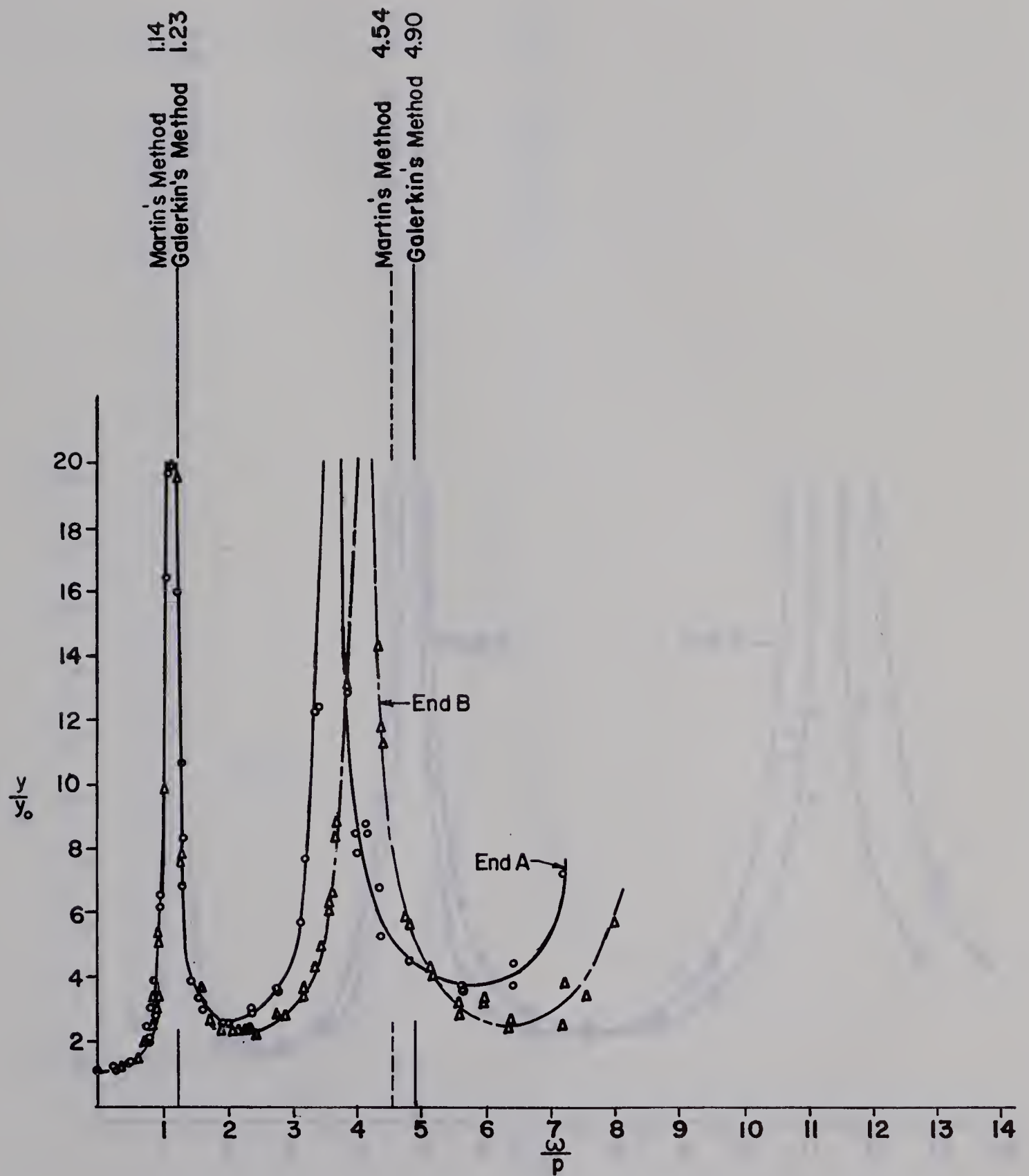


Figure 3-6 Frequency Spectrum for an 8" Aluminum Beam $\alpha=0.0$, $B=0.8$, $p=125.9$ cps

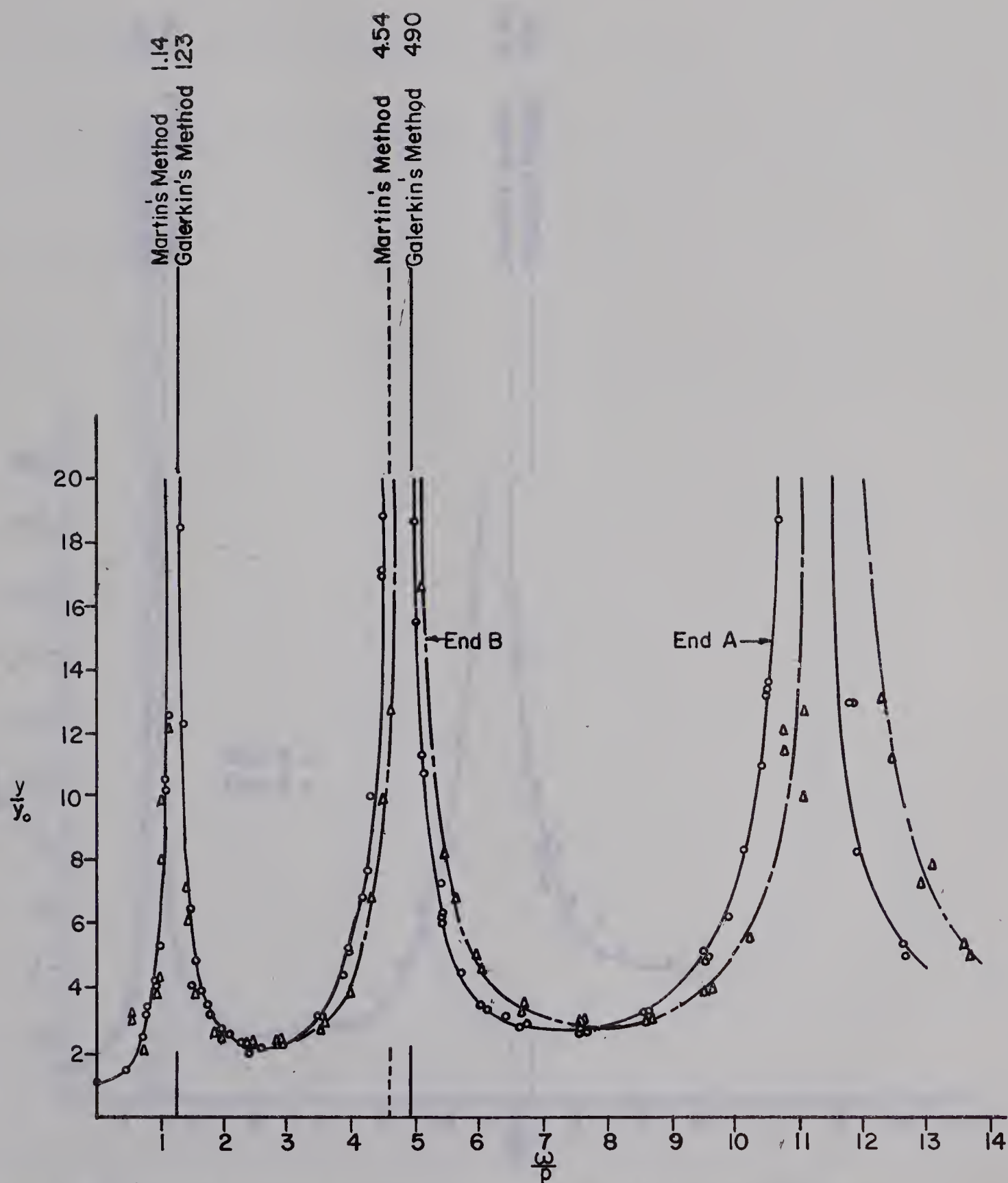


Figure 3-7 Frequency Spectrum for a 16" Aluminum Beam $\alpha=0.0$, $\beta=0.8$, $p=31.5$ cps

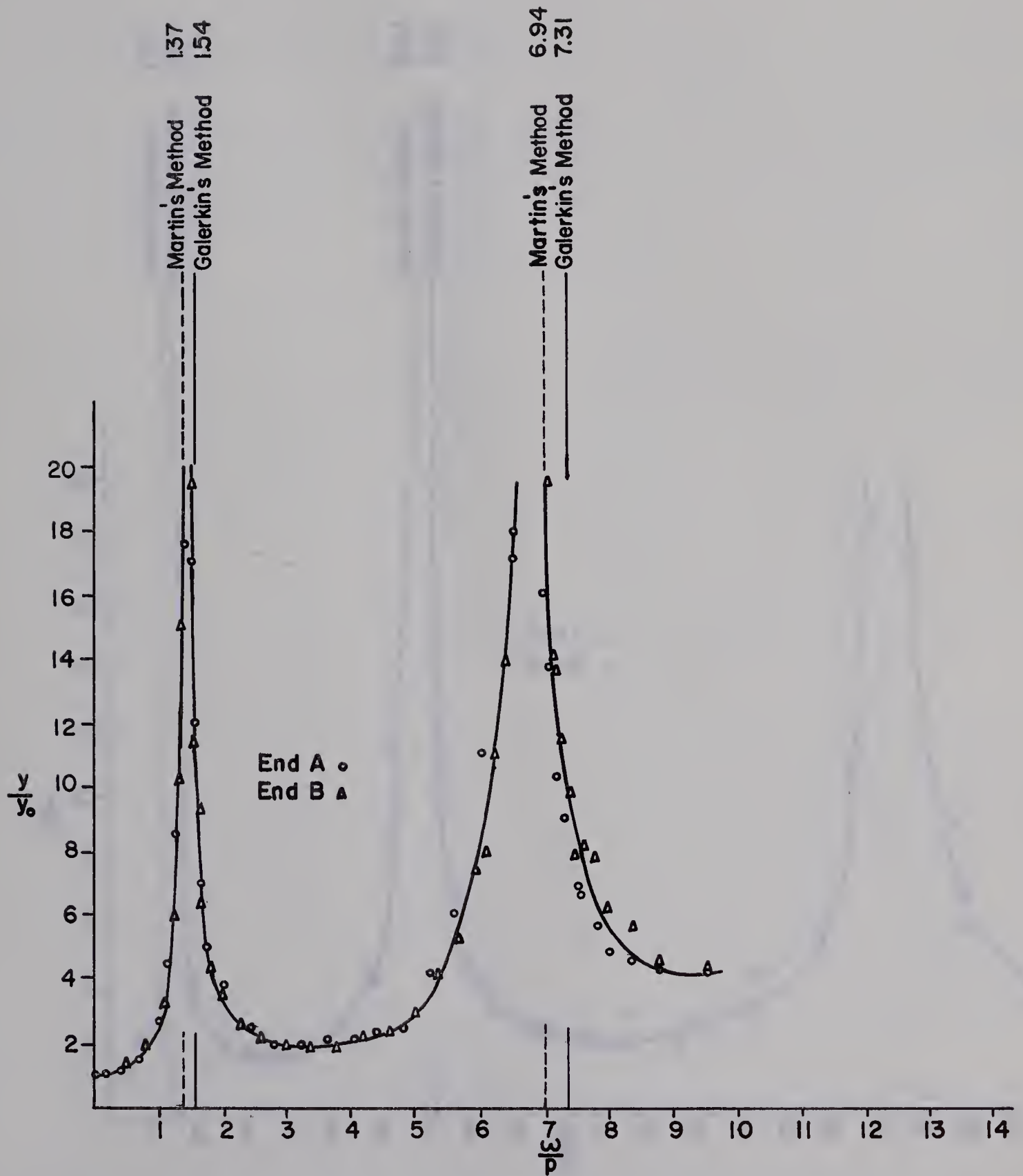


Figure 3-8 Frequency Spectrum for an 8" Aluminum Beam $\alpha=0.8$, $\beta=0.0$, $p=125.9$ cps

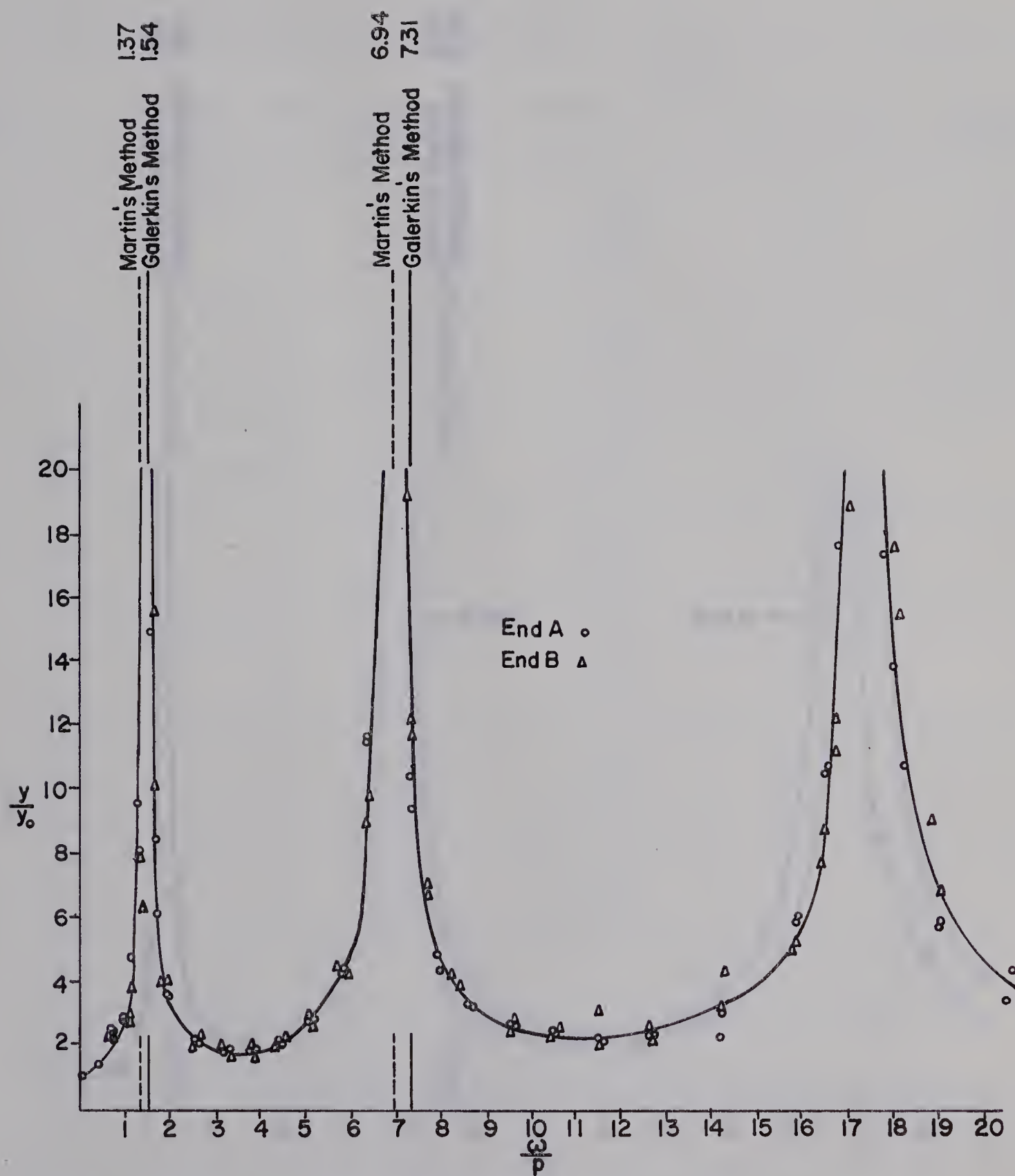


Figure 3-9 Frequency Spectrum for a 16" Aluminum Beam $\alpha=0.8$, $\beta=0.0$, $p=31.5$ cps

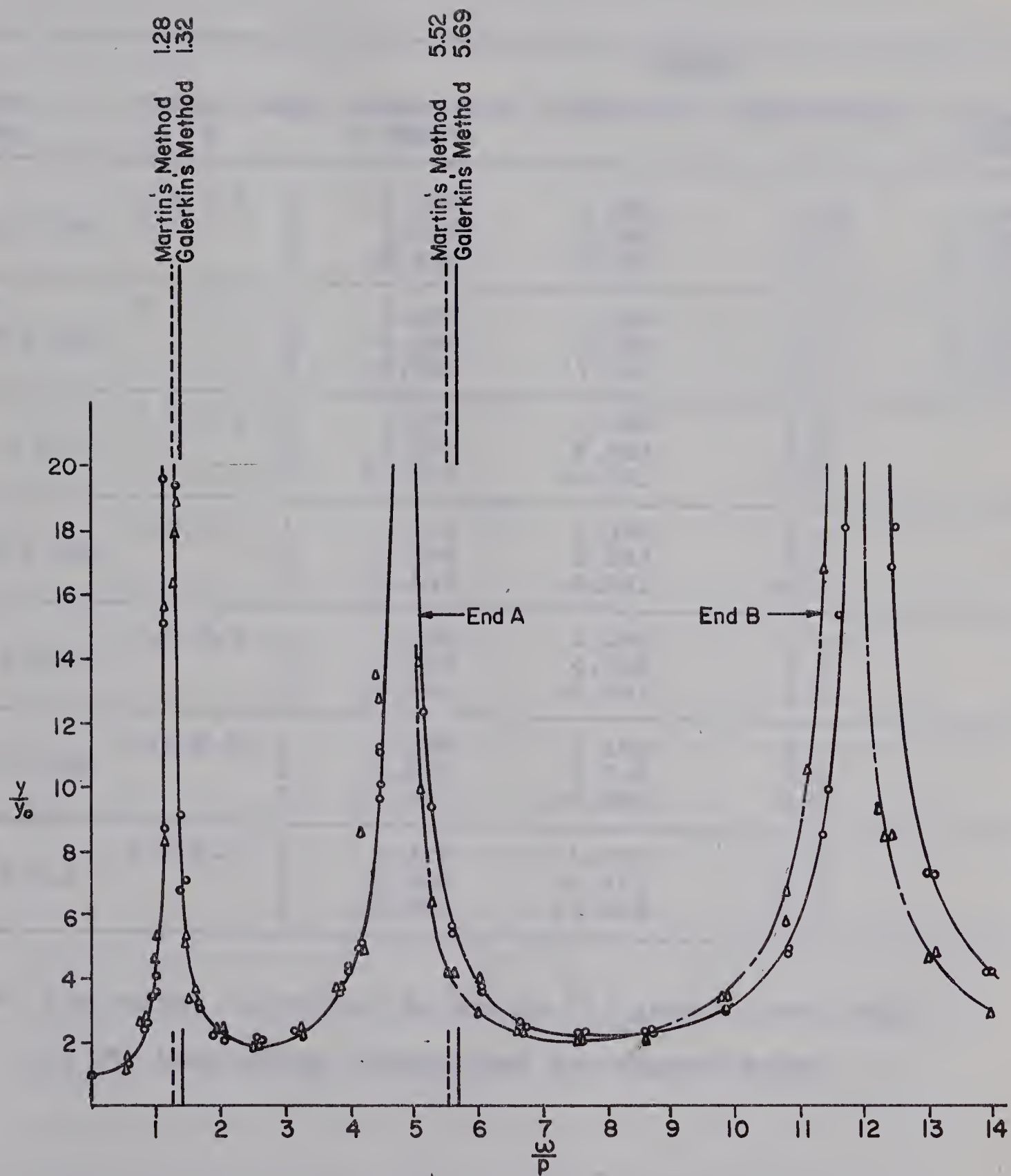


Figure 3-10 Frequency Spectrum for a 16" Aluminum Beam $\alpha=0.5$, $\beta=0.5$, $p=31.5$ cps

TABLE 3-2 EXPERIMENTAL VALUES FOR
FREQUENCY RATIOS - ω/p

LENGTH INCHES	TAPER		MODE	METHOD			
	α	β		GALERKIN'S 3 TERMS	MARTIN'S*	EXPERIMENTAL	CLOSED FORM
16 p=31.5 cps	0.0	0.0	1	1.000	1.000	0.95	1.000
			2	6.268	6.267	5.90	6.267
			3	18.846	17.551	16.4	17.551
8 p=125.9 cps	0.0	0.0	1	1.000	1.000	0.93	1.000
			2	6.268	6.267	5.5	6.267
			3	18.846	17.551	15.5	17.551
16 p=31.5 cps	0.8	0.0	1	1.535	1.368	1.5	
			2	7.307	6.994	6.8	
			3	21.155	18.222	17.5	
8 p=125.9 cps	0.8	0.0	1	1.535	1.368	1.4	
			2	7.307	6.944	6.7	
			3	21.155	18.222	16.6	
16 p=31.5 cps	0.0	0.8	1	1.226	1.140	1.2	
			2	4.897	4.538	4.8	
			3	17.615	10.892	11.5	
8 p=125.9 cps	0.0	0.8	1	1.226	1.140	1.1	
			2	4.897	4.538	3.8	
			3	17.615	10.892	8.7	
16 p=31.5 cps	0.5	0.5	1	1.316	1.277	1.2	
			2	5.693	5.551	4.7	
			3	18.788	13.865	11.9	

* The values calculated by Martin [5] give a lower bound to the theoretical frequencies for tapered beams.

with the vibrating beam. With the clamp deforming, the point where the boundary conditions, $y = 0$ and $\frac{dy}{dx} = 0$, were satisfied had to move to a distance greater than l from the free tip of the beam. For the case of an eight inch uniform beam, it was found that the experimental value for the fundamental frequency was $\omega/p = 0.93$, where p is given by

$$p = \sqrt{\frac{EI_0}{\rho A_0}} \left(\frac{1.875}{l} \right)^2. \quad (2.2.5)$$

If all the parameters, except l , in the formula for p are assumed to be exact, the error in p due to an error in the length is given by

$$\frac{dp}{p} = - \frac{2dl}{l}.$$

In order for this to give a fundamental frequency ratio of 0.93 instead of 1.00, dl must be equal to +0.28 inches. On the basis of the marks observed on the beam, a dl of 0.28 inches was indeed possible, and it was concluded that the deformation of the clamp was an important factor which could cause an error in the experimental frequency.

Another observation which was made is that the error between the experimental values and the theoretical values increased as the mode of vibration increased, and also that the range of frequencies around each mode for which the magnification factor was large increased for each higher mode (See Table 3-3). Since the amount of energy in the

system increased as the mode of vibration was increased, the amount of energy transferred to the clamp also had to increase. This would give larger clamp deflections and thereby increase the effective length of the beam. A second factor which contributed to the larger error at the higher modes was the increased effect of shearing force and rotary inertia, which also tend to lower the natural frequency of vibration.

From the fact that the range of frequencies for which the magnification factor was large increased as the mode of vibration increased, it is suspected that the damping present in the beams also increased with frequency. Since the effect of damping is to decrease the natural frequencies of vibration, the increases in the damping may also have contributed to the error that increased with the mode of vibration.

The next observation made was that for two tapered beams made to the same specifications, (end A and end B of the existing beams) the natural frequencies of vibration of the second and third modes varied as much as eight per cent between the two beams, with the largest differences being recorded for the beams with a thickness taper only. A careful examination of the beams revealed that the top and bottom surfaces were not plane. Since the thickness dimension of the beams was small (0.25 inches at the root and decreasing toward the free end), small variations in the

thickness dimension could have had a significant effect on the stiffness of the beams and ultimately effect the natural frequencies of vibration. The irregularity in the surfaces of the beams could possibly have contributed to the differences in the natural frequencies of two beams which were made to the same specifications.

The next observation made was that for magnification factors of greater than $y/y_0 = 7.0$, the tip deflections were unstable. This was indicated by the scatter of the points plotted on the frequency spectrums where the magnification factors are greater than 7.0. When the beams were vibrating at a frequency near a natural frequency, the variations in the tip deflections were plainly visible.

In Table 3-3 the range of frequencies around each mode for which the magnification factors were greater than 6.0 are given. This table illustrates the fact that the range of frequencies around each mode for which the magnification factors were large increased as the mode of vibration increased. It can also be seen from this table that the frequencies which were calculated for the first and second modes using the three term trial solution in Galerkin's method are usually in the range where the experimentally obtained magnification factors are greater than 6.0. The values which were computed for the third mode are a first approximation to the third natural frequency, and could be improved if more terms were taken in the trial solution.

TABLE 3-3 RANGE OF FREQUENCIES FOR WHICH THE
MAGNIFICATION FACTORS WERE GREATER THAN 6.0

TAPER		MODE	RANGE OF ω/p FOR $y/y_0 > 6.0$		CALCULATED
α	β	1	8" BEAM	16" BEAM	ω/p
0.0	0.0	1	0.80 - 1.00	0.75 - 1.10	1.000
		2	5.09 - 6.01	5.5 - 6.3	6.288
		3	-	15.7 - 17.2	18.846
0.8	0.0	1	1.20 - 1.67	1.2 - 1.7	1.535
		2	5.8 - 8.0	6.1 - 7.7	7.307
		3	-	16.3 - 18.9	21.155
0.0	0.8	1	0.8 - 1.3	0.95 - 1.4	1.226
		2	3.1 - 4.7	4.1 - 5.7	4.897
		3	-	9.8 - 13.4	17.615
0.5	0.5	1	-	1.1 - 1.4	1.316
		2	-	4.2 - 5.5	5.693
		3	-	10.7 - 13.3	18.788

CHAPTER IV

CONCLUSIONS AND DISCUSSION

4.1 Conclusions

The values which were obtained from Galerkin's method for the natural frequencies of vibration are an upper bound to the frequencies which were obtained from the closed form solutions. This can be seen in the results which are presented in Chapter III.

For linearly tapered cantilever beams, this method converged very well to the fundamental frequency of vibration, for all values of width and thickness taper, when the three term trial solution was used. When the thickness taper was less than $\beta = 0.5$ the two term trial solution gave acceptable values for the fundamental frequency. The three term trial solution gave acceptable values for the second frequency of vibration for all values of α when β was less than 0.8. For values of β greater than 0.8 the results began to diverge and the error increased as β approached 1.0.

By comparing the frequencies obtained from Galerkin's method to the frequencies obtained from the closed form solutions, it was found that the values obtained for the third natural frequency were 7.7 per cent high for $\alpha = 0.0$ and $\beta = 0.0$, and 11.6 per cent high for the case $\alpha = 1.0$,

$\beta = 0.0$. For the cases where $\beta = 1.0$ the frequencies obtained were over 100 per cent high. These errors could be reduced if more terms were used in the trial solutions. However it should be emphasized that a degree of difficulty was encountered in solving the determinant which was obtained when the three term trial solution was used. The difficulty encountered in this case was finding the differences between large numbers when the differences were very small. This condition would probably get worse if more terms were used in the trial solution. With this fact in mind, it was concluded that Galerkin's method is not suitable for finding the frequencies of vibration of modes higher than the third mode when the trial solution for the deflected shape is of the form

$$y = a_1(6x^2 - 4x^3 + x^4) + a_2(20x^2 - 10x^3 + x^5) + a_3(45x^2 - 20x^3 + x^6) + \dots$$

From the values obtained for the second natural frequency of vibration of linearly tapered beams, it was noted that for beams of the same width taper, the decrease in the frequency of the second mode was very nearly linear as β was increased from 0.0 to 0.5. When these linear sections were extrapolated to $\beta = 1.0$ for $\alpha = 0.0$ and $\alpha = 1.0$, the frequencies obtained for the second mode agreed very well with the frequencies which were obtained from the closed form solutions. On this basis it was concluded that the frequency

of the second mode of vibration is nearly linearly dependent on the thickness taper on the beam. This is discussed further in Section 4.3.

It was found that the effects of shearing force and rotatory inertia have a negligible effect on the fundamental frequency of vibrations of uniform beams. When the frequencies of the second and third modes were calculated, the influence of these effects was found to increase as the mode of vibration was increased and as the length of the beams decreased. When Galerkin's method was applied to the equation of motion for a wedge when the effects of shearing force and rotatory inertia were considered, it was found that tapering the beam magnified the effects of shearing force and rotatory inertia on the fundamental frequency of vibration.

When the values obtained for the fundamental frequency of vibration of a wedge that were obtained when shearing force and rotatory inertia were considered were compared to the frequencies obtained from the closed form solutions, and the experimental frequencies, it was found that for a beam with $\alpha = 0.0$ and $\beta = 1.0$, the frequency obtained was 27.5 per cent lower than the frequency obtained from the closed form solution, and that the experimental frequencies were higher than the theoretical frequencies. On the basis of these facts it was concluded that the values obtained from Galerkin's method for this case were in error. This is

possibly due to the round off errors that were present when the frequency equation was being solved.

The experimental results obtained in the form of frequency spectrums show that the range of frequencies around each mode for which the magnification factors were large increased as the mode of vibration increased. This indicates that the damping present is a function of the frequency. The increasing damping partially explains the larger errors between the theoretical results and the experimental results at the higher modes of vibration.

4.2 Summary of Conclusions

1. The natural frequencies of vibration of linearly tapered beams obtained using Galerkin's method are higher than the experimental frequencies and the frequencies obtained from the closed form solutions.
2. For values of β less than 0.5, Galerkin's method gives acceptable values for the frequency of the second mode when three terms are used in the trial solution.
3. With the trial solution which was used, Galerkin's method is not suitable for calculating the natural frequencies of modes higher than the third mode.
4. The natural frequencies of vibration of the second mode decrease almost linearly as the thickness taper is increased.

5. When the effects of shearing force and rotatory inertia are considered, Galerkin's method does not give satisfactory values for the fundamental frequency of vibration.

4.3 Second Mode Frequencies

When the values obtained using the three term trial solution were compared to the values obtained from the closed form solutions, it was noted that the frequencies of vibration of the second mode were within 0.1 per cent for beams of constant thickness and linearly varying width. It was also found that for beams of the same width taper the second natural frequency decreased almost linearly as the thickness taper was increased from $\beta = 0.0$ to $\beta = 0.5$, and when these linear sections were extrapolated to $\beta = 1.0$ for $\alpha = 0.0$ and $\alpha = 1.0$ the values given for the second natural frequencies were within five per cent on the frequencies given by the closed form solutions. This led to the conclusion that the frequency of the second mode of vibration decreases linearly as the thickness taper is increased.

When the second natural frequencies which were computed from the closed form solutions (See Table 4-1) were compared, it was found that for beams with $\alpha = 0.0$ the ratio of the frequency when $\beta = 1.0$ to the frequency when $\beta = 0.0$ was 0.691, and for beams with $\alpha = 1.0$ the ratio of the frequency

TABLE 4-1 SECOND MODE FREQUENCY RATIOS OBTAINED FROM
THE CLOSED FORM SOLUTIONS

BETA ALPHA	0.0	1.0
	0.0	1.0
0.0	6.267	4.326
1.0	8.830	6.016

when $\beta = 1.0$ to the frequency when $\beta = 0.0$ was 0.681. By assuming that the values of the corresponding ratios remain between the limits 0.681 and 0.691 for all values of α , and using the fact that the second natural frequency decreases linearly as β increases, it has been found that the frequency of the second mode of vibration can be estimated accurately for any combination of linear width and thickness taper once the frequency of vibration of the corresponding beam of constant thickness has been determined.

To find the frequencies it was assumed that for any fixed value of α , the ratio of the frequency when $\beta = 1.0$ to the frequency when $\beta = 0.0$ was 0.686. The frequency ratio for any value of α and β was then found from the relationship

$$\frac{\omega}{p} = \left(\frac{\omega}{p} \right)_0 (1 - 0.314\beta)$$

where $\frac{\omega}{p}$ is the frequency ratio for a beam with the desired

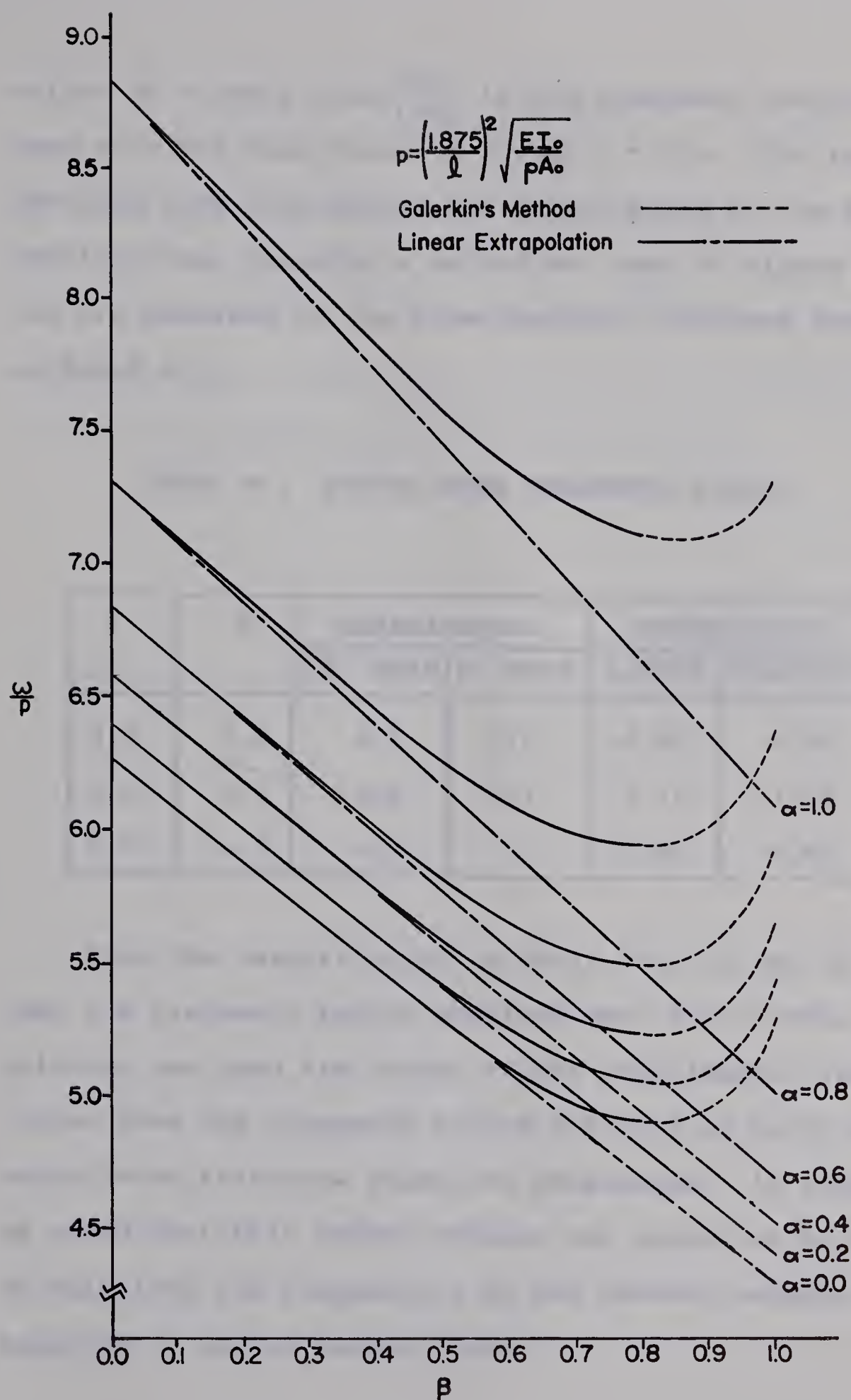


Figure 4-1 Second Natural Frequency of Vibration Versus Thickness Taper

values of α and β , and $\left(\frac{\omega}{p}\right)_0$ is the frequency ratio for a beam with the same value of α when $\beta = 0.0$. The results obtained from this method are superimposed on the results obtained when Galerkin's method was used in Figure 4-1, and are compared to the experimentally obtained frequencies in Table 4-2.

TABLE 4-2 SECOND MODE FREQUENCY RATIOS

α	β	EXPERIMENTAL		THEORETICAL	
		16" BEAM	8" BEAM	LINEAR	GALERKIN'S
0.0	0.8	4.8	3.8	4.68	4.90
0.8	0.0	6.8	6.7	7.31	7.31
0.5	0.5	4.7	---	5.65	5.69

From the results given in Table 4-2, it can be seen that the frequency ratios obtained when this linear extrapolation was used are closer to the experimental frequency ratios than the frequency ratios obtained by using Galerkin's method when thickness taper was considered. It should also be noted that this method reduces the amount of work required to calculate the frequencies of the second, because the equation of motion reduces from

$$E \left[I'''Y''' + 2I''Y'''' + IY^{IV} \right] = -\rho A \ddot{Y}$$

for beams with tapers in both directions to

$$E \left[2I''Y'''' + IY^{IV} \right] = -\rho A \ddot{Y}$$

for beams with a width taper only, and the expressions for the area and moment of inertia become linear.

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APPENDIX I

DERIVATION OF THE EQUATION OF MOTION FOR A WEDGE WITH
SHEARING FORCE AND ROTATORY INERTIA CONSIDERED.

For the wedge shown in Figure I-1,

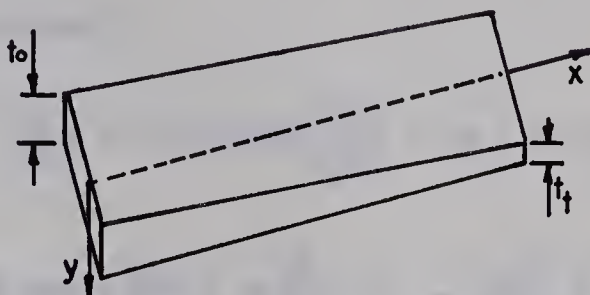


FIG. I-1 WEDGE

$$t = t_0 \left(1 - \frac{\beta x}{l} \right),$$

where $\beta = \frac{t_0 - t_t}{t_0},$

$$A = A_0 (1 - \beta x),$$

and $I = I_0 (1 - \beta x)^3, \text{ where } X = \frac{x}{l}.$

When shearing force and rotatory inertia are considered the equations which govern the motion are:

$$\frac{d}{dx} \left[EI \frac{d\psi}{dx} \right] + k' AG \left(\frac{dy}{dx} - \psi \right) = -\rho \omega^2 I \psi \quad (2.2.9)$$

$$\frac{d}{dx} \left[k' AG \left(\frac{dy}{dx} - \psi \right) \right] = -\rho \omega^2 A y \quad (2.2.10)$$

Expressing A as

$$A = \frac{A_0}{I_0 (1-\beta X)^2} I$$

and defining

$$D = \frac{k' G A_0}{I_0} ,$$

the equations become

$$\frac{d}{dx} \left[E(1-\beta X)^3 \frac{d\psi}{dx} \right] + D(1-\beta X) \left(\frac{dy}{dx} - \psi \right) = -\rho \omega^2 (1-\beta X)^3 \psi \quad (I.1)$$

$$\frac{d}{dx} \left[(1-\beta X) \left(\frac{dy}{dx} - \psi \right) \right] = -\frac{\rho \omega^2}{k' G} (1-\beta X) y \quad (I.2)$$

Let $(1-\beta X) = u$,

then

$$\begin{aligned} \frac{d}{dx} &= \frac{d}{du} \frac{du}{dx} \\ &= -\frac{\beta}{\ell} \frac{d}{du} \\ &= -c \frac{d}{du} . \end{aligned}$$

Substitute u into equations I.1 and I.2 and differentiate I.1 with respect to u to get

$$\begin{aligned} c^2 E \left[6u\psi' + 6u^2\psi'' + u^3\psi''' \right] - D \frac{d}{du} \left[u(cy' + \psi) \right] \\ = -\rho \omega^2 \left[3u^2\psi + u^3\psi' \right] , \end{aligned} \quad (I.3)$$

$$\frac{d}{du} \left[u(cy' + \psi) \right] = -\frac{\rho \omega^2}{ck' G} uy , \quad (I.4)$$

where the primes represent differentiation with respect to u.

Substituting the right hand side of I.4 into I.3 gives

$$c^2 E \left[6u\psi' + 6u^2\psi'' + u^3\psi''' \right] + \frac{D\rho\omega^2}{ck G} uy = -\rho\omega^2 \left[3u^2\psi + u^3\psi' \right] \quad (I.5)$$

Elimination of ψ from

$$\left[6u\psi' + 6u^2\psi'' + u^3\psi''' \right] \quad (A)$$

$$\frac{d}{du} (I.4) = 2(cy'' + \psi') + u(cy''' + \psi'') = -\frac{\rho\omega^2}{ck G} (uy' + y)$$

$$2\psi' = - \left[2cy'' + u(cy''' + \psi'') + \frac{\rho\omega^2}{ck G} (uy' + y) \right] \quad (I.6)$$

$\frac{d^2}{du^2} (I.4)$ gives

$$u\psi'''' = \left[-3(cy''' + \psi'') + ucy^{IV} + \frac{\rho\omega^2}{ck G} (uy'' + 2y') \right] \quad (I.7)$$

Substitution of I.6 and I.7 into (A) gives a term which is free from ψ . Equation I.5 then becomes

$$c^3 E \left[u^2 y^{IV} + 6uy'''' + 6y'' + \frac{\rho\omega^2}{c^2 k G} \left[u^2 y'' + 5uy' + 3y \right] \right] - \frac{D\rho\omega^2}{ck G} y = \rho\omega^2 \left[3u\psi + u^2\psi' \right] \quad (I.8)$$

Elimination of ψ from

$$\left[3u\psi + u^2\psi' \right] \quad (B)$$

Substituting u into I.1 gives

$$c^2 E \left[3u \psi' + u^2 \psi'' \right] - D(cy' + \psi) = -\rho\omega^2 u^2 \psi. \quad (I.9)$$

from which ψ can be found in terms of y and derivatives of ψ .

$$\psi = \frac{c^2 E \left[3u \psi' + u^2 \psi'' \right] - Dcy'}{(D - \rho\omega^2 u^2)} \quad (I.10)$$

Re-arranging I.4 gives

$$-\psi = cy' + cuy'' + u\psi' + \frac{\rho\omega^2}{ck G} uy. \quad (I.11)$$

Differentiating I.10 and collecting terms gives

$$\begin{aligned} & \psi' \left[D^2 - 2D\rho\omega^2 u^2 + \rho^2 \omega^4 u^4 - 3DEc^2 - 3Ec^2 \rho\omega^2 u^2 \right] \\ &= \psi'' \left[5DEc^2 u - 3Ec^2 \rho\omega^2 u^3 \right] + \psi''' \left[DEc^2 u^2 - Ec^2 \rho\omega^2 u^4 \right] \\ &- Dc \left[(d - \rho\omega^2 u^2) y'' + 2\rho\omega^2 uy' \right] \end{aligned} \quad (I.12)$$

$\frac{d}{du}$ (I.11) gives

$$-\psi' = 2cy'' + cuy''' + u\psi'' + \psi' + \frac{\rho\omega^2}{ck G} (uy' + y) \quad (I.13)$$

$\frac{d^2}{du^2}$ (I.11) gives, after re-arranging the terms

$$-u\psi'' = 3cy''' + cuy^{IV} + 3\psi'' + \frac{\rho\omega^2}{ck G} (uy'' + 2y') \quad (I.14)$$

Substitute I.13 and I.14 into I.12.

$$\begin{aligned}
\psi' = & \left[-DEc^3 \left[u^2 y^{IV} + 5uy'''' + 4y''' + \frac{\rho\omega^2}{c^2 k' G} (2y + 4uy' + u^2 y'') \right] \right. \\
& + Ec^3 \rho\omega^2 u^3 \left[uy^{IV} + 3y'''' + \frac{\rho\omega^2}{c^2 k' G} (uy'' + 2y') \right] \\
& \left. - Dc \left[(D - \rho\omega^2 u^2) y'' + 2\rho\omega^2 uy' \right] \right] / \\
& \left[(D - \rho\omega^2 u^2)^2 + DEc^2 - 3Ec^2 \rho\omega^2 u^2 \right]. \tag{I.15}
\end{aligned}$$

Substitute I.11 and I.15 into (B) to get

$$\begin{aligned}
& \left[3u\psi + u^2 \psi' \right] \\
= & - \left[3cu^2 y'' + 3cuy' + \frac{3\rho\omega^2}{ck' G} u^2 y \right] \\
& + 2u^2 \left[-DEc^3 \left[u^2 y^{IV} + 5uy'''' + 4y''' + \frac{\rho\omega^2}{c^2 k' G} (u^2 y'' + 4uy' + 2y) \right] \right. \\
& + Ec^3 \rho\omega^2 u^3 \left[uy^{IV} + 3y'''' + \frac{\rho\omega^2}{c^2 k' G} (uy'' + 2y') \right] \\
& \left. - Dc \left[(D - \rho\omega^2 u^2) y'' + 2\rho\omega^2 uy' \right] \right] / \\
& \left[(D - \rho\omega^2 u^2)^2 + DEc^2 - 3Ec^2 \rho\omega^2 u^2 \right], \tag{I.16}
\end{aligned}$$

which is free from terms involving ψ and its derivatives.

To get an equation for y in terms of u , substitute I.16 into I.8 and collect the terms. After dividing through by the common factor $(D - 3\rho\omega^2 u^2)$ the equation becomes,

$$\begin{aligned}
& \frac{d^4 y}{du^4} \left[Ec^3 u^2 (D - \rho \omega^2 u^2 + Ec^2) \right] \\
& + \frac{d^3 y}{du^3} \left[Ec^3 u (6D - 4\rho \omega^2 u^2 + 6Ec^2) \right] \\
& + \frac{d^2 y}{du^2} \left[(D + Ec^2) (6Ec^3 + c\rho \omega^2 u^2) + \frac{Ec\rho \omega^2 u^2}{k' G} (D + Ec^2) \right. \\
& \quad \left. - \frac{Ec\rho^2 \omega^4 u^4}{k' G} - c\rho^2 \omega^4 u^4 \right] \\
& + \frac{dy}{du} \left[(D + Ec^2) (3c\rho \omega^2 u + \frac{5Ec\rho \omega^2 u}{k' G}) - \frac{3Ec\rho^2 \omega^4 u^3}{k' G} - c\rho^2 \omega^4 u^3 \right] \\
& + y \left[\left(\frac{D + Ec^2}{ck' G} \right) (3Ec^2 \rho \omega^2 - D\rho \omega^2 + 2\rho^2 \omega^4 u^2) - \frac{\rho^3 \omega^6 u^4}{ck' G} \right] = 0 \quad (I.17)
\end{aligned}$$

When u is replaced by $(1-\beta X)$, the equation of motion for a linearly tapered wedge with shearing force and rotatory inertia becomes

$$\begin{aligned}
& \frac{1}{cl^4} \frac{d^4 y}{dX^4} \left(E(1-\beta X)^2 (D - \rho \omega^2 (1-\beta X)^2 + Ec^2) \right) \\
& - \frac{1}{l^3} \frac{d^3 y}{dX^3} \left(E(1-\beta X) (6D - 4\rho \omega^2 (1-\beta X)^3 + 6Ec^2) \right) \\
& + \frac{1}{cl^2} \frac{d^2 y}{dX^2} \left((D + Ec^2) (6Ec^2 + \rho \omega^2 (1-\beta X)^2) - \rho^2 \omega^4 (1-\beta X)^4 \right. \\
& \quad \left. + \frac{D + Ec^2}{k' G} E\rho \omega^2 (1-\beta X)^2 - \frac{E\rho^2 \omega^4}{k' G} (1-\beta X)^4 \right)
\end{aligned}$$

$$\begin{aligned}
& - \frac{1}{\ell} \frac{dy}{dx} \left((D + Ec^2) \left(3\rho\omega^2(1-\beta X) + \frac{5E\rho\omega^2(1-\beta X)}{k'G} \right) - \frac{3E\rho^2\omega^4(1-\beta X)^3}{k'G} \right. \\
& \quad \left. - \rho^2\omega^4(1-\beta X)^3 \right) \\
& + y \left(\left(\frac{D + Ec^2}{ck'G} \right) \left(3Ec^2\rho\omega^2 - D\rho\omega^2 + 2\rho^2\omega^4(1-\beta X)^2 \right) - \frac{\rho^3\omega^6}{ck'G} (1-\beta X)^4 \right) \\
& = 0, \tag{I.18}
\end{aligned}$$

where $X = \frac{x}{\ell}$.

APPENDIX I-B

APPLICATION OF GALERKIN'S METHOD TO THE EQUATION OF MOTION
FOR A WEDGE WHEN SHEARING FORCE AND ROTATORY INERTIA ARE
CONSIDERED.

By multiplying Equation I.18 by c and collecting the coefficients of the powers of ω the equation of motion becomes

$$\begin{aligned}
& A_1(1-\beta X)^4 y \omega^6 \\
& + \left[A_2(1-\beta X)^4 y'' + A_3(1-\beta X)^3 y' + A_4(1-\beta X)^2 y \right] \omega^4 \\
& + \left[A_5(1-\beta X)^4 y^{IV} + A_6(1-\beta X)^3 y'''' + A_7(1-\beta X)^2 y'' + A_8(1-\beta X) y' \right. \\
& \left. + A_9 y \right] \omega^2 + \left[A_{10}(1-\beta X)^2 y^{IV} + A_{11}(1-\beta X) y'''' + A_{12} y'' \right] = 0 \tag{2.3.13}
\end{aligned}$$

where primes denote differentiation with respect to X ,

$$X = \frac{x}{\ell} \quad \text{and}$$

$$\begin{aligned} A_1 &= \frac{-\rho^3}{k' G} & A_7 &= \frac{A \left(\rho + \frac{E\rho}{k' G} \right)}{\ell^2} \\ A_2 &= - \left(\frac{\rho^2 + \frac{E\rho^2}{k' G}}{\ell^2} \right) & A_8 &= \frac{-cA \left(3\rho + \frac{5E\rho}{k' G} \right)}{\ell} \\ A_3 &= \frac{c \left(\frac{3E\rho^2}{k' G} + \rho^2 \right)}{\ell} & A_9 &= \frac{A}{k' G} \left(3Ec^2\rho - D\rho \right) \\ A_4 &= \frac{2\rho^2 A}{k' G} & A_{10} &= \frac{EA}{\ell^4} \\ A_5 &= \frac{-E\rho}{\ell^4} & A_{11} &= - \frac{6EAc}{\ell^3} \\ A_6 &= \frac{4Ec\rho}{\ell^3} & A_{12} &= \frac{6Ec^2 A}{\ell^2} \\ A &= D + Ec^2 . \end{aligned}$$

When the one term trial solution was used in the application of Galerkin's method to Equation 2.3.13 the equation

$$\int_{\ell}^L \left[y_1 \right] \phi_i dX = 0 , \quad (I.19)$$

was formed. Since

$$y_1(X) = a_1(6X^2 - 4X^3 + X^4)$$

$$\phi_i(X) = (6X^2 - 4X^3 + X^4) ,$$

and the derivatives of y were taken with respect to X
Equation I.19 became

$$\begin{aligned}
 & \int_0^1 \left[A_1 (1-\beta X)^4 a_1 (6X^2 - 4X^3 + X^4) \omega^6 \right. \\
 & + \left[A_2 (1-\beta X)^4 12a_1 (1-2X+X^2) + A_3 (1-\beta X)^3 a_1 (12X-12X^2+4X^3) \right. \\
 & + \left. A_4 (1-\beta X)^2 a_1 (6X^2-4X^3+X^4) \right] \omega^4 \\
 & + \left[A_5 (1-\beta X)^4 24a_1 + A_6 (1-\beta X)^3 (-24a_1)(1-X) + A_7 (1-\beta X)^2 12a_1 (1-2X+X^2) \right. \\
 & + \left. A_8 (1-\beta X) a_1 (12X-12X^2+4X^3) + A_9 a_1 (6X^2-4X^3+X^4) \right] \omega^2 \\
 & + \left[A_{10} (1-\beta X)^2 24a_1 + A_{11} (1-\beta X) (-24a_1)(1-X) \right. \\
 & + \left. A_{12} 12a_1 (1-2X+X^2) \right] \left. \right] (6X^2-4X^3+X^4) dX = 0 . \tag{I.20}
 \end{aligned}$$

From Equation I.20 it can be seen that the constant a_1 is a common factor which can be cancelled, leaving a cubic equation in ω^2 . When the roots of Equation I.20 were found, only one root turned out to be greater than zero, and that value was taken to be the square of the fundamental frequency of vibration.

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